6(b). Combinatorial Optimization via LPs

Lecturer: Prahladh Harsha *Scribe: Yadu Vasudev*

In this lecture, we look at Yannakakis'[Yan91] approach to relating the minimum size of LPs for polytopes to a combinatorial parameter and some connections to communication complexity.

6(b).1 Introduction

The flavor of a combinatorial optimization problem is to maximize an objective function over a set of valid points. In the case of TSP, the valid set of points $S \subseteq \{0,1\}^{\binom{n}{2}}$ is the set of valid tours on the graph and the objective function is the minimum weight tour. In the cases that we will be interested in, the objective function will be linear.

To make the problem amenable to linear programming, we look at the convex hull of S, which we will denote by $\text{conv}(S)$. The caveat here is that the polytope thus formed will have large number of facets and thus will be exponentially big to even represent. The question that we will be addressing in this lecture is, for what sets S will the polytope corresponding to $\text{conv}(S)$ have a small number of facets? We note that even if $\text{conv}(S)$ has exponentially many facets, it can still have an efficient LP based algorithm as long as there is a seperation oracle.

Definition 6(b).1. *For a polytope P in $\mathbb{R}^n$, we say that the polytope P' in $\mathbb{R}^{n+m}$* expresses *P if and only if $P = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m,\ (x,y) \in P'\}$.*

Hence, given a polytope with a large number of facets in $\mathbb{R}^n$ if we are able to construct a polytope P' in $\mathbb{R}^{n+m}$ with a only a polynomial number of facets which expresses P, then we can apply the techniques of LP to this new polytope to get the optimum.

6(b).1.1 Parity polytope

As an example, lets look at the parity polytope $PP = \text{conv}(\{x \in \{0,1\}^n \mid x \text{ has an even number of 1s}\})$. The parity polytope is thus given by the following constraints.

$$\begin{aligned} \sum_{i\in S} x_i - \sum_{i\notin S} x_i &\le |S| - 1,\ \forall \text{ even sized sets } S \\ 0 \le x_i &\le 1,\ \forall i \in [n] \end{aligned} \tag{6(b).1.1}$$

This polytope has 2^n many constraints, one for each subset of $[n]$. We now construct a polytope with larger number of variables but only polynomially many constraints. Observe that $PP = \text{conv}(\bigcup_{k\ odd} \text{conv}(\{x \in \{0,1\}^n \mid \text{number of ones in } x \text{ is } k\}))$. This gives us a new polytope with a few extra variables

$$\begin{aligned} \sum_k \alpha_k &= 1 \\ x_i &= \sum_{k\ odd} z_{ik} \\ \sum_i z_{ik} &= k\alpha_k \end{aligned} \tag{6(b).1.2}$$

This construction of the new polytope works for any symmetric function. Thus we have the following corollary.

Corollary 6(b).2. *For any symmetric function f, there exists a polynomial sized polytope P'_f which expresses the polytope $P_f = \mathrm{conv}(\{x \in \{0,1\}^n \mid f(x) = 1\})$*

In the rest of the lecture, we look at what other predicates have this property. To that end, we will look at a combinatorial parameter related to the polytope.

6(b).2 The slack matrix

Consider a polytope P given by the system of inequalities $Cx \leq d$. The slack matrix SM of this polytope P is an $M \times N$ matrix where M is the number of constraints of the matrix and N, the number of extreme points. We will denote the i^{th} constraint by c^i and the j^{th} extreme point by u^j. The $(i,j)^{th}$ entry of the slack matrix is given by $SM_{i,j} = d_i - \langle c^i, u^j \rangle$. Thus $SM_{i,j} \geq 0$ for all i, j.

Definition 6(b).3. *The smallest m such that there exist non-negative matrices F, V, $M \times m$ and $m \times N$ respectively such that $SM = FV$ is the positive rank of SM.*

We will denote it by $\mathrm{rank}^+(SM)$. We now state the theorem due to Yannakakis [Yan91] which characterizes the expressibility of a polytope with another with larger number of variables but lesser contraints.

Theorem 6(b).4. *A polytope P with n variables is expressible by a polytope P' with $O(n+m^*)$ number of constraints if and only if $m^* = \mathrm{rank}^+(SM)$.*

Proof. ($\Leftarrow$) Let P be given by the set of inequalities $Cx \leq d$ where C is an $M \times n$ matrix and d is an $n \times 1$ matrix. Since $\mathrm{rank}^+(SM) = m^*$, there exists F, V such that $SM = FV$. We will denote the i^{th} row of F by F_i and the j^{th} columns of V by V_j. Then,

$$SM_{i,j} = \langle F_i, V_j \rangle = d_i - \langle c^i, u^j \rangle.$$

Define the polytope P' to be given by the constraints

$$Cx + Fy = d \ ; \ y \geq 0.$$

Every extreme point u^j in the polytope P has a correspondig point (u^j, v^j) in P'. The number of constraints defining the polytope P' is $O(n + m^*)$ since there are only so many variables and all the constraints are equality constraints.

($\Rightarrow$) Let the polytope P' express P with $O(m^* + n)$ constraints and variables. We will assume that P' is described by the polytope

$$\begin{aligned} C'x + D'y &\leq d' \\ x &\geq 0 \\ y &\geq 0 \end{aligned} \tag{6(b).2.1}$$

We will add a few new variables to convert all the inequalities to equalities. The new constraints are given by

$$Rx + Sy = t \tag{6(b).2.2}$$

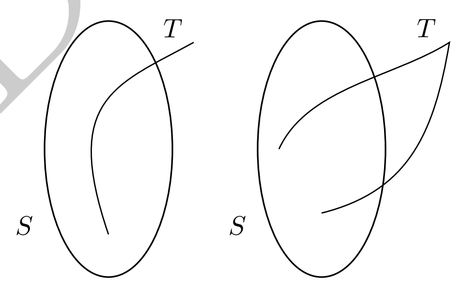
Since P' expresses P, for each vertex u^j of P, there exists a y^j which satisfies (6(b).2.2). For every constraint $C_i x$, there is a vector μ_i such that $\mu_i R = C_i$, $\mu_i t = d_i$ and $\mu_i S = F_i \geq 0$. Thus for the solution (u^j, y^j) corresponding to the vertex u^j, $C_i u^j + F_i y^j = \mu_i \left[R \ S\right] \left[u^j \ y^j\right]^T = \mu_i t = d_i$. Thus, the $(i,j)^{th}$ entry of the slack matrix is $F_i y^j$. Hence, we can factorize $SM = FV$ where the rows of F are the vectors F_i and the columns of V are the vectors y^j. $\square$

6(b).2.1 The spanning tree polytope

For a graph $G(V, E)$, the spanning tree is a tree on the vertices V of G that covers all the vertices. The spanning tree polytope $ST(G)$ is given by

$$\begin{aligned} &\sum_{i,j \in S} x_{ij} \leq |S| - 1, \ \forall \text{ sets } S \subseteq V \\ &\sum_{i,j} x_{ij} = n - 1 \\ &0 \leq x_{ij} \leq 1 \end{aligned} \tag{6(b).2.3}$$

This polytope has exponentially many facets, one for each subset of vertices. The slack matrix for this polytope has rows indexed by the subsets of the vertex set and the columns indexed by the trees in the graph. $SM(S,T)$ is the number of nodes in S whose parent in T is not in S. This is explained by the following diagram.



Tight constraint Slack constraint

This gives us the following decomposition of the slack matrix into F and V where the columns of F and the rows of V are indexed by 3-tuples (k, i, j). For the matrix F,

$$F(S,(k,i,j)) = \begin{cases} 1 & \text{if } k, i \in S, j \notin S \\ 0 & \text{otherwise} \end{cases} \tag{6(b).2.4}$$

and

$$V((k,i,j),T) = \begin{cases} 1 & \text{if } j \text{ is the parent of } i \text{ at the tree rooted at } k. \\ 0 & \text{otherwise} \end{cases} \tag{6(b).2.5}$$

Thus, the slack matrix for the spanning tree polytope has positive rank $O(n^3)$. This gives us the polytope $ST'(G)$ which has only polynomially many facets.

$$\begin{aligned} x_{ij} &= \lambda_{kij} + \lambda_{kji} \\ \lambda_{kkj} &= 0 \\ \sum_j \lambda_{kij} &= 1, \ \forall i \neq k \in [n] \\ \lambda_{kij} &\geq 0 \\ 0 \leq x_{ij} &\leq 1 \end{aligned} \tag{6(b).2.6}$$

where for a tree T, λ_{kij} is 1 if T is rooted at k and j is the parent of i in T.

6(b).3 Connection to communication complexity

Consider the following communication game. For a polytope P, Alice is given a constraint defining the polytope and Bob is given a vertex of the polytope. Alice and Bob should output a 1 if the constraint given to Alice is not tight for the vertex given to Bob. The communication matrix for this game M_{SM} is same to the slack matrix for the polytope SM for entries which are zero. All non-zero entries in the slack matrix are replaced by 1s in the communication matrix.

Let F, V be the non-negative matrices such that $SM = FV$. Let $\text{rank}^+(SM) = m^*$. We will now see a connection between the communication complexity of this game and the positive rank of the slack matrix.

Claim 6(b).5. *$SM_{ij} \geq 0$ if and only if $\exists\ k \in [m^*]$ such that $F_{ik} > 0$ and $V_{kj} > 0$.*

This is true since both F and V are non-negative matrices and the only way for $SM_{ij} > 0$ is when $F_{ik} > 0$ and $V_{kj} > 0$. As a corollary we get

Corollary 6(b).6. $\mathrm{N}(M_{SM}) \leq O(\log m^*)$

The converse direction does not hold in the case of non-deterministic protocols whereas for deterministic communication complexity, we have the following observation.

Observation 6(b).7. *Let $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be such that $\mathrm{D}(f) \leq c$. Then $\text{rank}^+(M_f) \leq 2^c$.*

Proof. Consider the following matrices F and V. The rows of F are indexed by the inputs to Alice and the columns are indexed by the accepting transcripts. The rows of V are indexed by the accepting transcripts for Bob and the columns are indexed by his inputs. An entry $F(x, \tau)$ is 1 if the transcript τ is consistent with the input x and zero otherwise. Similarly for $V(\tau, y)$. By the construction we can see that $M_f = FV$. Thus $\text{rank}^+(M_f) \leq 2^c$ if $\text{D}(f) \leq c$. □

6(b).3.1 Vertex packing polytope

For a graph $G(V, E)$, the vertex packing polytope $VP(G)$ is given by

$$VP(G) = \text{conv}(\left\{x \in \{0,1\}^{|V(G)|} \mid x \text{ is the characteristic vector of an independent set in } G\right\}).$$

A complete description of this polytope, identifying the various facets is not known. The set $\left\{x \in \{0,1\}^{|V(G)|} \mid x \text{ is the characteristic vector of an independent set in } G\right\}$ can be decribed by the integer linear program

$$\begin{aligned} x_i + x_j \leq 1,\ (i,j) \in E \\ x_i \in \{0,1\} \end{aligned} \tag{6(b).3.1}$$

Notice that just relaxing the condition $x_1 \in \{0,1\}$ to $0 \leq x_i \leq 1$ does not give the convex hull of the set. This can be seen by looking at $G = K_3$, the complete graph on 3 vertices. The independent set for G is the set $\{(1,0,0),(0,1,0),(0,0,1)\}$, but $(1/2,1/2,1/2)$ satisfies the relaxation we mentioned earlier even though it is not in the convex hull of the independent sets. As far as expressing the vertex packing poytope by polytopes with polynomially many facets is concerned, we can observe the following fact.

Observation 6(b).8. *If for all graphs G, $VP(G)$ is expressible by polynomial sized polytopes, then* $\mathsf{NP} \subseteq \mathsf{P}/poly$.

We could modify (6(b).3.1) to get the following polytope

$$\begin{aligned} & x_i + x_j \leq 1,\ (i,j) \in E \\ & \sum_{i \in K} x_i \leq 1,\ \text{for all cliques } K \\ & 0 \leq x_i \leq 1 \end{aligned} \tag{6(b).3.2}$$

This exactly characterizes the independent sets for a large class of graphs known as perfect graphs. G is a perfect graph if any induced subgraph H satisfy the condition that $\chi(H)$ is the size of the larges clique in H. Comparability graph of a poset is an example of a perfect graph. The comparability graph $G(V, E)$ of a poset $(P, \leq)$ has $V = P$ and $(i, j) \in E$ if either $i \leq j$ or $j \leq i$.

The slack matrix, SM for this polytope has its rows indexed by the cliques in the graph and the columns indexed by the independent sets. $SM(C, I) = 1$ if the clique C and the independent set I intersect and is zero otherwise. This also corresponds to the communication problem *clique-independent set* $CLIS_G$, where Alice is given a clique C from G and Bob is given an independent set I. At the end of the protocol they have to answer if there is a vertex that is present in both the clique and the independent set. The communication matrix of this problem is same as the slack matrix of (6(b).3.2).

The non-deterministic complexity of the *clique-independent set* problem is open. It is known that $\mathrm{D}(CLIS_G) = O(\log^2 n)$ (refer problem 4 in problem set 1). Thus there exists a polytope P' with $O(n^{\log n})$ constraints that expresses $VP(G)$ for perfect graphs.

References

[Yan91] Mihalis Yannakakis. *Expressing combinatorial optimization problems by linear programs.* J. Computer and System Sciences, 43(3):441–466, 1991. (Preliminary Version in *20th STOC*, 1988). doi:10.1016/0022-0000(91)90024-Y.