Problem Set 4

- Due Date: **6 Dec, 2017**
- Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Refering sources other than the text book and class notes is strongly discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side.
- Problems 1–6 from O'Donnell's book/course.
- Be clear in your writing.

1. **[Orthogonal decomposition]**

   Given any $f : \{1,1\}^n \to \mathbb{R}$, consider $f^S : \{1,1\}^n \to \mathbb{R}$ defined by $f^S := \widehat{f}(S)\chi_S$. We have $f = \sum_{S \subseteq [n]} f^S$ as functions, and this orthogonal decomposition has the following three properties:

   i $f^S(x)$ depends only on the coordinates of x in S;

   ii $\mathbb{E}_x[f^S(x) f^T(x)] = 0$ if $S \neq T$;

   iii $\sum_{T \subseteq S} f^T$, denoted $f^{\subseteq S}$, gives the conditional expectation of f conditioned on the coordinates in S.

   (a) Prove property item 1iii; i.e., $f^{\subseteq S} = \mathbb{E}[f_{x \to S}]$, where the expectation is over the bits in $\overline{S} = [n] \setminus S$. (Here the notation is that $x \in \{1,1\}^n$, but in the expression $f_{x \to S}$, we only restrict the S-coordinates of f using the S-bits of x; the $\overline{S}$-bits of x are ignored.)

   In the rest of this problem we establish the same kind of decomposition for general real-valued functions on product probability spaces. Specifically, let Ω be any finite set and let π be a probability distribution on Ω. We think of the n-fold product set Ω^n as having the product probability distribution induced by π. All $\Pr[], \mathbb{E}[]$ in what follows refer to this product distribution.

   (b) We first make property item 1iii hold by fiat: For $S \subseteq [n]$, we define $f^{\subseteq S} : \Omega^n \to \mathbb{R}$ to be the function depending only on the coordinates in S giving the conditional expectation; i.e., $f^{\subseteq S}(x) := \mathbb{E}[f_{xS}]$, where the expectation is over the product probability distribution on the coordinates outside S. Now given this definition, explicitly write how we should define the functions f^S so that the equations $f^{\subseteq S} = \sum_{T \subseteq S} f^T$ hold. Check also that property item 1i holds with your definitions.

   [Hint: inclusion-exclusion.]

   (c) Show that $\mathbb{E}[f^{\subseteq S}(x) f^{\subseteq T}(x)] = \mathbb{E}[f^{\subseteq (S \cap T)}(x)^2]$, straight from our definition of $f^{\subseteq S}$.

(d) Now show property item 1ii, that $\mathbb{E}[f^S(x)f^T(x)] = 0$ when $S \neq T$.

Remark: This orthogonal decomposition of functions f is often a good substitute for Fourier analysis when the domain is a product probability space other than $\{1, 1\}^n$.

2. **[Coarse Thresholds]** Exercises 8.27 and 8.28 in O'Donnell's book.
3. **[Tightness of hypercontractivity theorem]** Exercise 9.3 in O'Donnell's book.
4. **[hypercontractivity for sparse functions]** Exercise 9.34 in O'Donnell's book.
5. **[reverse hypercontractivity]** Exercise 10.6 in O'Donnell's book.
6. **[log-Sobolev inequality]** Exercise 10.23 in O'Donnell's book.