Problem Set 1

- Due Date: **21 Feb, 2019**
- Turn in your problem sets electronically (LaTeX, pdf or text file) by email. If you submit handwritten solutions, start each problem on a fresh page.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Refering sources other than class notes and given references is discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side.
- Be clear in your writing.

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1. **[Trace from spectral decomposition]** (3+(3+3)) = 9

   The trace of a matrix $M \in \mathbb{R}^{n\times n}$ is defined as $\mathrm{Tr}(M) := \sum_{i\in[n]} M_{i,i}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of M (with multiplicities).

   (a) Using the fact that the λ_i's are roots of the characteristic polynomial $\det(xI - M) = 0$ (with multiplicities), conclude that $\mathrm{Tr}(M) = \sum \lambda_i$.

   (b) In this part, we will use the spectral decomposition to give an alternate proof of the above fact (for the case of real symmetric matrices).

      i. Prove that for any two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

      ii. Using the above part and the spectral decomposition for symmetric matrices (i.e., for every real symmetric M, there exists an orthonormal matrix Φ and diagonal matrix Λ such that $M = \Phi\Lambda\Phi^T$) conclude that for any real symmetric M, we have $\mathrm{Tr}(M) = \sum \lambda_i$.

2. **[top eigenvector can be made strictly positive]** (5+5) =10

   Let A be the adjacency matrix of a connected weighted graph G (with non-negative edge weights).

   (a) Suppose φ is an eigen-vector of A with all non-negative entries. Show that φ has all strictly positive entries.

   (b) Let μ_1 be the largest eigenvalue of A. Show that there exists a corresponding eigenvector for μ_1 with all non-negative entries.

   The two parts together imply that there exists a top eigenvector with all strictly positive entries.

3. **[Colouring using top eigenvector]** (9+2) =11

   Let A be the adjacency matrix of an undirected graph G, and let φ be a top eigenvector with eigenvalue μ_1. Note that $\kappa \geq 0$ and (by the previous problem) φ can be so chosen as to have positive entries. Let us assume further that the vertices are ordered so that the entries of φ are arranged in non-increasing order (i.e., $\varphi_1 \geq \varphi_2 \geq \cdots \geq \varphi_n > 0$).

Consider now the following colouring procedure, which is based on the above order. We start with an empty list L of colours. We then process the vertices $u_1, u_2, \ldots, u_n$ in order, and for any given i, construct the set S of the colours assigned to neighbors u_j of u_i with $j < i$. If $S = L$, then we create a new color c, set $L \leftarrow L \cup \{c\}$ and assign the colour c to u_i. Otherwise, if $L \setminus S$ is non-empty, we choose a color from $L \setminus S$ (according to some pre-defined choice rule) and assign it to u_i. Note that this procedure produces a proper coloring of the graph.

How large can L can be at the end of the algorithm? Show that your bound is tight by giving an appropriate example.

4. **[$\nu_n \approx 2 \Leftrightarrow$ almost bipartite]** (6+14)=20

Let $G = (V, E)$ be an undirected graph and let $L = D - A$ be its Laplacian. The largest eigenvalue of the normalized Laplacian, denoted by ν_n satisfies

$$\nu_n = \max_{x \neq 0} \frac{x^T L x}{x^T D x}.$$

Recall that in class, we proved that $\nu_n \leq 2$ and that equality holds iff the graph G is bipartite.

(a) **[almost bipartite $\Rightarrow$ ν_n almost 2]**

Suppose the MAXCUT in G has normalized cost at least $1 - \varepsilon$. That is, there exists a cut $(S, V \setminus S)$ such $|\partial S| \geq (1 - \varepsilon)|E|$. Prove that there is a non-zero vector $x \in \mathbb{R}^V$ such that

$$x^T (D + A) x \leq 2\varepsilon \cdot x^T D x.$$

Hence, conclude that $\nu_n \geq 2(1 - \varepsilon)$.

(b) **[ν_n almost 2 $\Rightarrow$ almost bipartite]**

In this part, we will prove the following theorem.

Theorem. *Let $\nu_n \geq 2(1 - \varepsilon)$ or equivalently there exists a non-zero vector $x \in \mathbb{R}^V$ such that $x^T (D + A) x \leq 2\varepsilon \cdot x^T D x$. Then there exists non-zero vector $y \in \{-1, 0, 1\}^V$ such that*

$$\frac{\sum_{\{u,v\} \in E} |y_u + y_v|}{\sum_{u \in V} d_u |y_u|} \leq \sqrt{8\varepsilon}.$$

To this end, we define the following randomized process that constructs a random non-zero vector $Y \in \{-1, 0, 1\}^V$ given a non-zero vector $x \in \mathbb{R}^V$ satisfying $x^T (D + A) x \leq 2\varepsilon \cdot x^T D x$. Since this latter condition is scale-invariant, we may assume wlog. that $\max_u |x_u| = 1$ and let $u_* \in V$ such that $|x_{u_*}| = 1$.

- Pick a value t uniformly in $[0, 1]$.
- Define $Y \in \{-1, 0, 1\}^V$ as follows:

$$Y_u = \begin{cases} -1 & \text{if } x_u < -\sqrt{t}, \\ 1 & \text{if } x_u > \sqrt{t}, \\ 0 & \text{otherwise, i.e., } |x_u| \leq \sqrt{t}. \end{cases}$$

i. (2 points) Prove that $\Pr[\exists u \in V, Y_u \neq 0] = 1$.

ii. (3 points) Prove that $\mathbb{E}[|Y_u|] = x_u^2$ and $\mathbb{E}[|Y_u + Y_v|] \leq |x_u + x_v| \cdot (|x_u| + |x_v|)$.

iii. (6 points) Prove that $\mathbb{E}\left[\sum_{\{u,v\}\in E} |Y_u + Y_v|\right] \leq \sqrt{8\varepsilon} \cdot \mathbb{E}\left[\sum_u d_u |Y_u|\right]$.

[Hint: Cauchy-Schwarz Inequality.]

iv. (3 points) Hence, conclude that there exists a non-zero vector $y \in \{-1,0,1\}^V$ such that $\sum_{\{u,v\}\in E} |y_u + y_v| \leq \sqrt{8\varepsilon}. \cdot \sum_{u\in V} d_u |y_u|$.

Discussion. *It is known that G is connected if* $\nu_2 \neq 0$. *Or equivalently,* $\varphi(G) \neq 0$ *iff* $\nu_2 \neq 0$. *Cheeger's inequalities give a "quantitative strengthening" of this statement by showing that*

$$\sqrt{2\nu_2} \geq \varphi(G) \geq \nu_2/2.$$

This problem is similar in spirit but work with ν_n *and "bipartiteness" instead of* ν_2 *and "connectedness" respectively.*

Define the bipartiteness ratio number *of a graph G to be*

$$\beta(G) := \min_{y\in\{-1,0,1\}^V} \frac{\sum_{\{u,v\}\in E} |y_u + y_v|}{2d \sum_{u\in V} |y_u|},$$

which is equivalent to

$$\beta(G) = \min_{S\subseteq V, (L,R) \text{ partition of } S} \frac{2\partial(L,L) + 2\partial(R,R) + \partial(S, V \setminus S)}{d|S|},$$

Observe that $\beta(G) = 0$ *iff G is bipartite. It is easy to check that* $\beta(G) = 0$ *iff* $\nu_n = 2$. *Problems 4a–4b are a quantitative strengthening of this claim as they demonstrate that*

$$\sqrt{2(2-\nu_n)} \geq \beta(G) \geq \frac{1}{2} \cdot (2 - \nu_n).$$

These problems are due to a result by Luca Trevisan.