**Today**

- Space Complexity (Part II)
  - * TQBF : PSPACE-complete
  - * Savitch's Theorem
  - * logspace reductions, NL
  - * NL = co NL

CSS.203.1 Computational Complexity
– Lecture # 8 (10 Mar '21)
Instructor: Prahladh Harsha

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"... Games harder than puzzles ..." – Even & Tarjan

**Theorem: TQBF is PSPACE-complete**

Last time: TQBF ∈ PSPACE (TQBF ∈ SPACE($n^2$) can be improved to SPACE($n$))

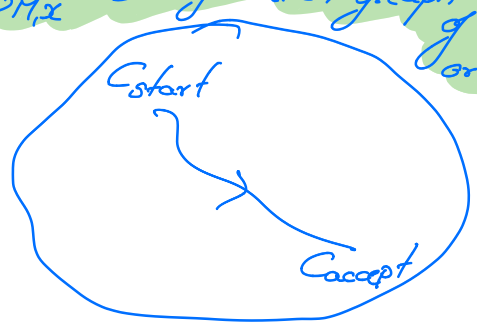
Today: TQBF is PSPACE-hard

ie, $\forall L \in$ PSPACE, $L \leq_p$ TQBF

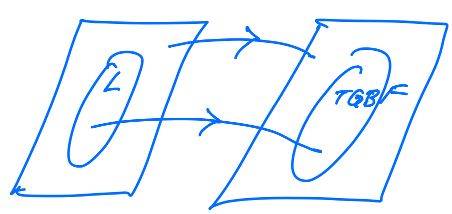
$L \in$ PSPACE

$L$ is accepted by some TM that runs in space $S(n)$ = polynomial in $n$.

$G_{M,x}$ : Configuration graph of $M$ on i/p $x$




$x \longmapsto \psi_x$




"$x$ is accepted by $M$"

$\Updownarrow$

$\exists$ a path from $C_{start} \rightsquigarrow C_{accept}$ in $G_{M,x}$

$$\varphi_M(C,C') = \begin{cases} 1 & \text{if } (C,C') \text{ is an edge in } G_{M,x} \\ 0 & \text{otherwise.} \end{cases}$$

$$\left.\begin{array}{l} \varphi_M : \{0,1\}^{2\cdot O(S)} \longrightarrow \{0,1\} \\ \Rightarrow \text{ time } O(S) \end{array}\right\} \rightarrow \text{Can do}$$

$$\psi_M(C,C') = \begin{cases} 1 & \text{if there exists a path from } C \text{ to } C' \text{ in } G_{M,x} \\ 0 & \text{otherwise} \end{cases}$$

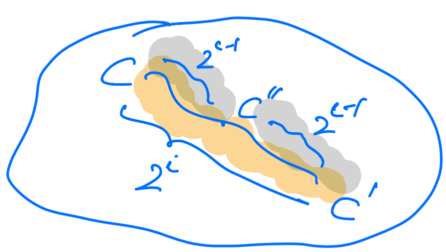
$$\psi_x :\triangleq \psi_M(C_{start}, C_{accept})$$

$$\psi_{M,i}(C,C') = \begin{cases} 1 & \text{if there exists a path of length} \leq 2^i \text{ from } C \text{ to } C' \text{ in } G_{M,x} \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi_M(C,C') \triangleq \psi_{M,O(S)}(C,C')$$

$$\psi_{M,0}(C,C') = \left.\left\{ \text{"} C \equiv C' \text{"} \right\} \text{ or } \left\{ \varphi_M(C,C') \right\}\right\} \rightarrow O(S)$$

$$\psi_{M,i}(C,C') = \exists C'' \ \psi_{M,i-1}(C,C'') \wedge \psi_{M,i-1}(C'',C')$$




Note: If implemented as above

$size(\psi_{M,i}) \leq 2 \cdot size(\psi_{M,i-1})$

$\rightarrow$ exponential blowup.

Use universal quantifier ($\forall$) to avoid this doubling

$$\psi_{M,i}(C,C') = \exists C'' \forall (D,D') \in \{(C,C''), (C'',C')\} \; \psi_{M,i-1}(D,D')$$

$$\text{time}(\psi_{M,i}) \leq \text{time}(\psi_{M,i-1}) + O(S)$$

Solving, $\text{time}(\psi_{M,i}) = O(i \cdot S)$

$$\text{time}(\psi_{M,O(S)}) = O(S^2)$$

Thus, we have an $O(S^2)$-time redn that maps $x \longmapsto \psi_x = \psi_M(C_{start}, C_{accept})$
$= \psi_{M,O(S)}(C_s, C_a)$.

Hence, $L \leq_p TQBF$. $\boxtimes$

<u>Observation:</u> $NPSPACE \leq_p TQBF$
(the exact same proof)

<u>Cor:</u> $NPSPACE = PSPACE$

(i.e, w.r.t space, non-determinism does not give you more than a polynomial advantage)

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What is the polynomial advantage
- $TQBF \in SPACE(n)$
- $L \in NSPACE(S(n))$, then $L \leq TQBF$ in time $O(S^2(n))$.

Putting the two together.

$L \in SPACE(S^2(n))$.

| | Space |
|---|---|
| Det space alg for $L$ | |
| - First reduce to TQBF | $S^2(n)$ |
| - Then solve TQBF | $S^2(n)$ |
| | $S^2(n)$ |

Can be improved to log space

## Savitch's Theorem

**Thm:** $NSPACE(S(n)) \subseteq SPACE(S^2(n))$. for space-constructible $S(n)$.

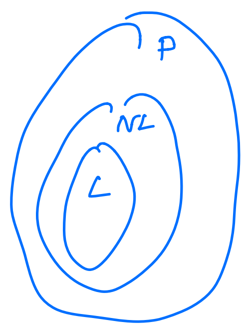
PSPACE-complete: TQBF

$$\exists x_1 \forall x_2 \exists x_3 \ldots \quad \varphi(x_1 \ldots x_n)$$

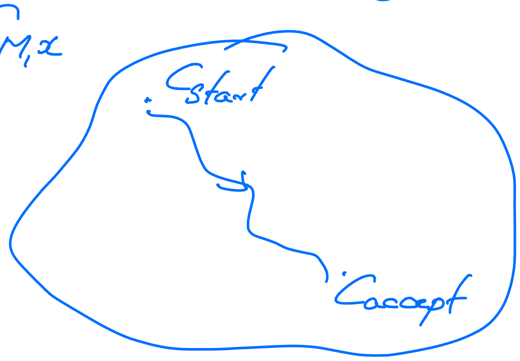
## Logspace

$L = SPACE(\log n)$

$NL = NSPACE(\log n)$.




$G_{M,x}$




# vertices is polynomial of $S(n)$

$= O(\log n)$

PATH = { (G, s, t) | G is a directed graph & there is a path from s to t in G }

**PATH ∈ NL**

Is PATH NL-complete?

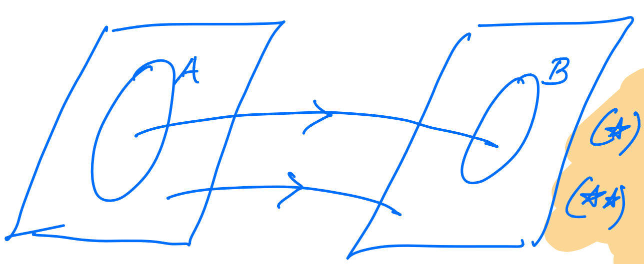
Certainly, $\forall A \in NL$, $A \leq_p PATH$ is } Not a very refined stmt since the redn is more complicated than the class.

Define logspace reduction

## Logspace Reductions:

A, B $\quad A \leq_L B$

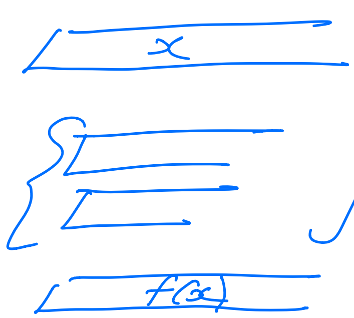
$f: \{0,1\}^* \to \{0,1\}^*$




(*) $x \in A \iff f(x) \in B$

(**) f is computable in logspace

f is computable in logspace




input tape / read only

work tape – logspace

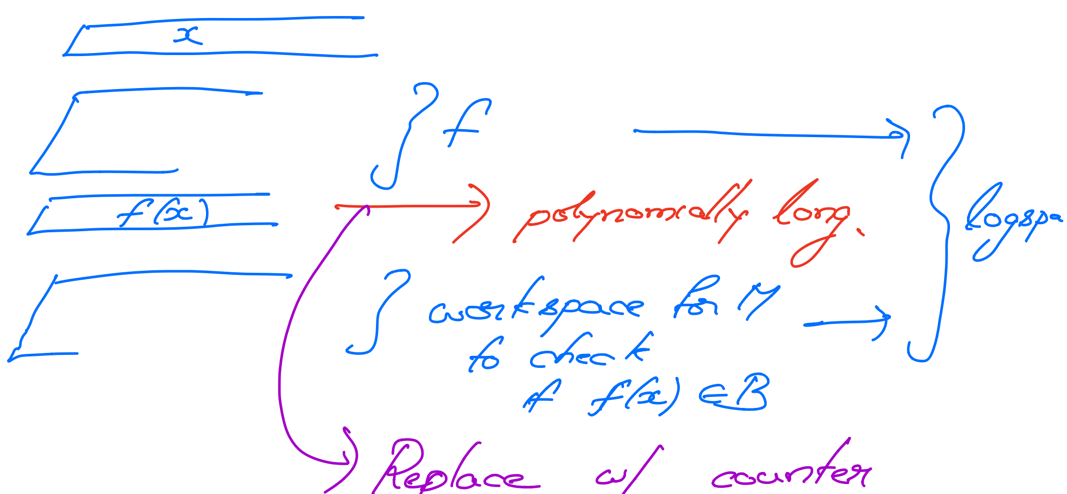
output tape / write once

# Properties of logspace reductions

1. $A \leq_L B$ & $B \in L \Rightarrow A \in L$
2. Transitivity: $A \leq_L B,\ B \leq_L C \overset{?}{\Rightarrow} A \leq_L C$

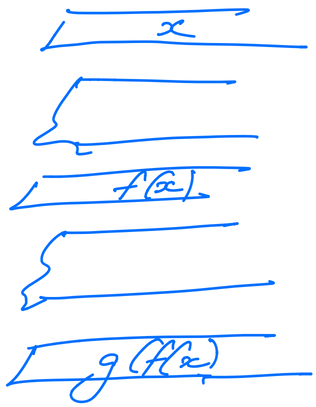
Proof: (1) Suppose $\underbrace{A \leq_L B}_{f}$ & $\underbrace{B \in L}_{M}$

Want to come up w/ a m/c N that solves A in logspace.




Idea: Don't run f & M in sequence
But start running M &
~~run~~ run f each time M needs to read an i/p bit.

(2) $\underbrace{A \leq_L B}_{f}$; $\underbrace{B \leq_L C}_{g} \Rightarrow A \leq_L C$




Don't do in sequence

Run $f$ as and when needed.

Return to the NL-completeness:

A is NL-hard if $\forall\, B \in NL,\ B \leq_L A$

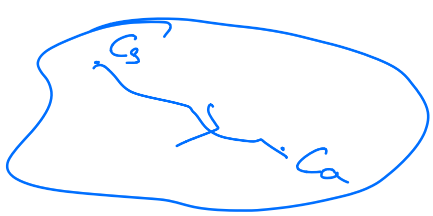
A is NL-complete if (i) A is NL-hard
(ii) $A \in NL$.

PATH : $PATH \in NL$.

Qn: Is PATH NL-hard.

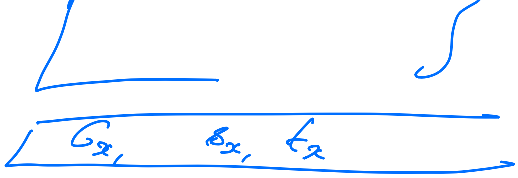
ie, $A \in NL$. $A \leq_L PATH$

$G_{M,x}$



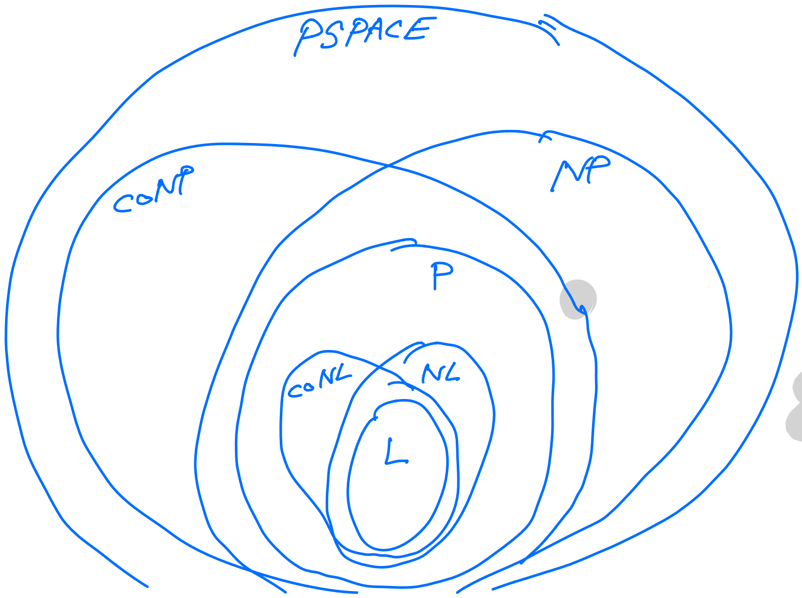

$x \mapsto (G_x, s_x, t_x)$

$G_x = G_{M,x}$
$s_x = C_{start}$
$t_x = C_{accept}$




~~Output the set of vertice~~
Output for every pair of vertices

the pair if they
are adjacent
config.




Next time:
coNL = NL