Today

- PCPs and hardness of approximation. (CFGLSS Reduction)

CS5.203.1 Computational Complexity

- Lecture #30 (2 Jun, 21)

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Recap:

PCP Theorem: $NP = PCP_{1,1/2}[O(\log n), O(1)]$.

Equivalence to inapproximability of MAX3SAT

gap_α-MAX3SAT* is NP hard.

Today:

Application towards the hardness of approximating the MAXCLIQUE

Gap_α-CLIQUE: Promise problem. $(\alpha \in (0,1))$

$YES = \{(G,k) \mid \exists$ a clique of size $\geq k$ in $G\}$

$NO = \{(G,k) \mid$ Every clique in G is of size $< \alpha k\}$

Goal: Does $\exists\, \alpha \in (0,1)$ s.t. there is a ptime reduction from SAT to gap_α-CLIQUE?

Thm [Feige-Goldwasser-Lovasz-Safra-Szegedy]

If $L \in PCP_{c,s}[\text{rand}=r;\ \text{query}=q,\ \text{time}=t]$ then there exist a deterministic $O(t \cdot 2^{r+q})$-time reduction from L to $\text{gap}_{s/c}$-CLIQUE.

PCP Theorem: $SAT \in PCP_{1,1/2}[C\log n, Q]$

PCP Theorem + FGLSS Thm:

$\exists$ a ptime reduction from SAT to $\text{gap}_{1/2}$-CLIQUE

Cor: $1/2$-approximating the MAX CLIQUE is NP-hard.

Proof of FGLSS Theorem.

Reduction: Karp reduction from SAT to MAXCLIQUE

$L \in PCP_{c,s}[r, q, t]$

Want a reduction R

$$L \longrightarrow gap_{s/c}\text{-CLIQUE}$$

$$x \longmapsto (G_x, k_x)$$

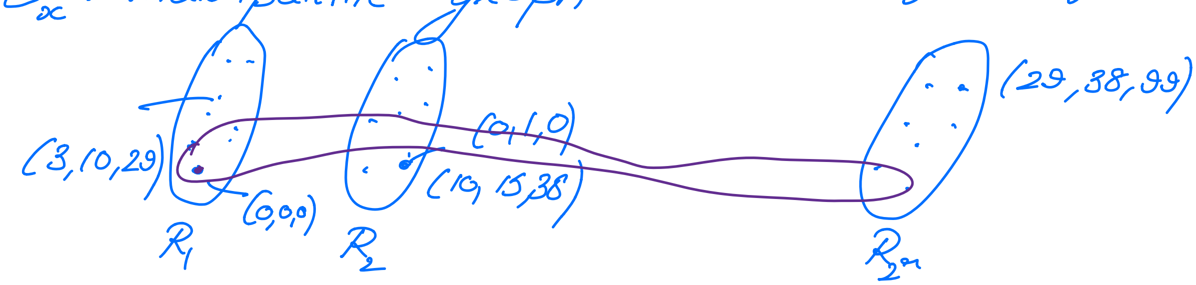
L has a (r, q, t)-restricted verifier. V

Comp: $x \in L \Rightarrow \exists \pi \ \Pr_R[V^\pi(x;R) = acc] \geq c$

Sound: $x \notin L \Rightarrow \forall \pi, \Pr_R[V^\pi(x;R) = acc] < s.$

x - instance of L.

G_x: multipartite graph $\quad V_x = \{0,1\}^r \times \{0,1\}^q$



$E_x : (R_1, (b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(q)})) \sim (R_2, (b_2^{(1)}, \ldots, b_2^{(q)}))$

if the following 2 condns are met

(i) $\forall i \in [2]$, $\bar{b}_i$ - satisfying assign to the predicate D_{R_i} based on randomness R_i

(ii) The 2 local views are consistent.

$k_x = c \cdot 2^r$

$x \mapsto (G_x, k_x)$

Running time of reduction: $2^{q+r} \cdot t$

Completeness:

$x \in L \Rightarrow \exists \pi, \Pr_R [V^\pi (x; R) = acc] \geq c$

$$S_\pi = \{ (R, \bar{b}) \mid R \in \{0,1\}^r ; (Q, D) \leftarrow V(x; R) \\ D(\bar{b}) = acc, \pi|_Q = \bar{b} \}$$

S_π - clique., $|S_\pi| \geq c \cdot 2^r$

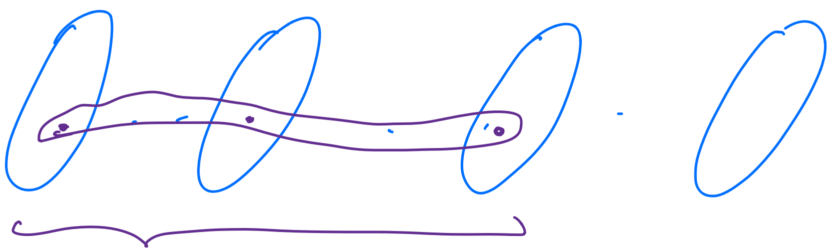
$(G_x, k_x) \in YES$ $\boxtimes$

Soundness:

Suppose $(G_x, k_x) \notin NO$

ie, $\exists$ a clique S of size

$$\alpha \cdot k_x = \frac{s}{c} \cdot c \cdot 2^q = s \cdot 2^q.$$



Given S clique, we can construct a proof π st

$$S \subseteq S_\pi$$

Then this proof π satisfies. s-fraction of the random coin tosses

$\Downarrow$

$x \in L$

(ie, $x \notin L \Rightarrow (G_x, k_x) \in NO$)

$\boxtimes$

Original Karp-reduction proving NP-completeness of clique

$\to$ Approximation preserving reduction

– Putting everything together

PCP Thm: $SAT \in PCP_{1,1/2}[c\log n, 3]$ for some constant $c \geq 1$

FGLSS Thm: $L \in PCP_{c,s}[r, q, t]$

$\Downarrow$

$L \leq_p gap_{s/c}\text{-CLIQUE}$ in time $O(t \cdot 2^{r+q})$

Cor: NP-hard to approximate clique to a factor better than 1/2.

– Repeat verifier twice.

$PCP_{1,s}[r, q, t] \subseteq PCP_{1,s^2}[2r, 2q, 2t]$

k-sequential repetition.

$PCP_{1,s}[r, q, t] \subseteq PCP_{1,s^k}[kr, kq, kt]$

In particular

$SAT \in PCP_{1,1/2^k}[kc\log n, 3k, poly]$

Cor: $\forall$ constant $\alpha \in (0,1)$, gap_α-MAXCLIQUE is NP-hard.

Sequential Repetition

– not using independent random coins

$k \cdot c \log n$

Recycling Randomness

– but using a k-step walk on spectral expander of size 2^r.

$c \log n + O(k)$

$$PCP_{1,\frac{1}{2}}[r, q, t] \subseteq PCP_{1,\frac{1}{2^k}}[r+O(k), O(kq), O(kt)]$$

Now, we can make k as large as $\Theta(\log n)$

Cor: $\exists \delta \in (0,1)$, gap_{1/n^δ}-CLIQUE is NP-hard.

Håstad: [Recyle Queries]
$\forall \varepsilon \in (0,1)$
$gap_{1/n^{1-\varepsilon}}$-CLIQUE is NP-hard
under randomized reductions

Zuckerman: under deterministic red$\underline{\underline{n}}$s

Amortized Query complexity

- What does each additional query buy you in soundness?

(each additional query reduces soundness by 1/2 in the limit)

Khot's conjecture: PCP Theorem on steroids

Unique Game Conjecture
- # queries = 2
- predicate (unique predicate) $ax + by = c$
- completeness - $1-\varepsilon$
- soundness - δ $\forall \delta, \varepsilon$
- alphabet (not necessarily Boolean)

Khot, Khot-Regev, Khot-Kindler-Mossel-Olieskwicz.

- VC is UG-hard to approx better than 1/2.
- MAXCUT is UG-hard to approx better than α_{GW}.

3-color.

gap version

YES = $\{G \mid G$ is 3-colorable$\}$

NO = $\{G \mid G$ requires M colors$\}$

$\cup$ |

NO' = $\{G \mid \alpha(G) \le \frac{1}{M}|V(G)|\}$

[Hås] $\forall \varepsilon \in (0,1)$. NP-hard to approx MAX3SAT $\frac{7}{8}+\varepsilon$ (even when the input is satisfiable)