Problem Set 2

- Due Date: **24 Mar, 2021**
- Turn in your problem sets electronically (LaTeX, pdf or text file) on Acadly.
- Collaboration is encouraged, but all writeups must be done individually and must include names of all collaborators.
- Referring to sources other than the text book and class notes is strongly discouraged. But if you do use an external source (eg., other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- The points for each problem are indicated on the side. The total for this set is **100** points.
- Be clear in your writing.

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1. Prove that $\mathsf{NTIME}(n) \neq \mathsf{P}$. (15)

   [Hint: You may not be able to show containment in either direction; just that the two classes are different! It may be helpful to use the time hierarchy theorems.]

2. **[Unique-witness]** (15)

   Show that U3COL $\in \mathsf{P}^{\mathrm{SAT}}$ where U3COL is defined as follows.

   $$\mathrm{U3COL} = \{G = (V, E) \mid G \text{ has a unique 3-coloring (up to permutation of color labels)}\}.$$

3. **[Baker-Gill-Solovay for NP vs coNP]** (15)

   Show that there is a language A relative to which $\mathsf{NP}^A \neq \mathsf{coNP}^A$.

   [Hint: It might be useful to remember that a coNP machine rejects if *any* of its non-deterministic paths reject and suitably modify the construction we did in class.]

4. **[PSPACE problems]** (10+15)

   (a) Antakshari: Let Σ be a finite alphabet and let $S = \{s \mid s \in \Sigma^*\}$ be a set of strings over Σ. For $s = s_1 s_2 \ldots s_n \in \Sigma^*$, let suff($s$) and pre($s$) be s_1 and s_n, respectively.

   Let P_0 and P_1 be two players. The game of Antakshari is played as follows: The game begins with P_0 choosing a string s_0 from S. The game then proceeds in rounds with player $P_{i \pmod 2}$ playing in round i as follows: During round i, player $P_{1 \pmod 2}$ picks a string s_i from S with the following property: $\mathrm{pre}(s_i) = \mathrm{suff}(s_{i-1})$, and for all $0 \leq j < i$, $s_i \neq s_j$, where s_j denotes the string picked in round j. A player is said to lose if she is not able to pick a string (i.e. no such string exists and she is stuck). Player P_0 is said to have a winning strategy on starting with s_0 if player P_0 wins the game irrespective of the subsequent moves of player P_1. Else, player P_1 is said to have a winning strategy.

   We define

   $$\text{Antakshari} = \{(S, s) \mid P_0 \text{ has a winning strategy on starting with } s\}$$

   Prove that Antakshari is in PSPACE.

(b) Let $\mathrm{EQ}_{\mathrm{RegExp}} = \{\langle R, S\rangle \mid R \text{ and } S \text{ are equivalent regular expressions}\}$. Show that $\mathrm{EQ}_{\mathrm{RegExp}} \in \mathsf{PSPACE}$.

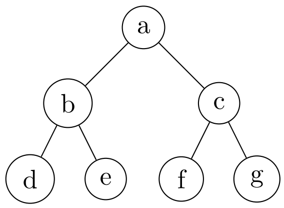
[Hint: It would be easier to show that $\mathrm{EQ}_{\mathrm{RegExp}} \in \mathsf{coNPSPACE}$ first.]

(For extra credit, show that Antakshari (and $\mathrm{EQ}_{\mathrm{RegExp}}$)) are in fact PSPACE-complete.)

5. **[Boolean Formula Evaluation]** (10+10)

(a) Prove that computing the DFS order (the order of vertices visited, including repetitions, in a DFS traversal that starts at the root and ends at the root) of an *undirected binary tree* $T = (V, E)$ can be done in L (logspace).

For example, the DFS order of the tree below is $a, b, d, b, e, b, a, c, f, c, g, c, a$.



[Hint: (a) You may assume that the tree is described as follows: For every vertex $v \in V$, there is a function $next_v : V \to V \cup \{\bot\}$ which gives a clockwise ordering of the edges around the vertex V. I.e., For every vertex v, there is a cyclic ordering among the neighbours of v and $next_v(u)$ is the next neighbour in this cyclic ordering if u is a neighbour of v and $\bot$ otherwise. Finally, check that one can make this assumption without loss of generality.]

(b) A Boolean formula φ on n inputs is a directed tree with n *sources* (vertices with no incoming edges) and one *sink* (vertex with no outgoing edges). All nonsource vertices are called *gates* and are labeled with one of $\vee, \wedge$ or $\neg$. The vertices labeled with $\vee$ or $\wedge$ have fan-in 2 and the vertices labeled with $\neg$ have fan-in 1. Let $x \in \{0,1\}^n$ be some input. The output of φ on x, denoted by $\varphi(x)$, is defined in the natural way. The Boolean formula evaluation problem deals with, given a formula φ on n inputs and $x \in \{0,1\}^n$, computing the value of $\varphi(x)$. Show that formula evaluation can be done in logspace. More precisely, define

$$\mathrm{FVAL} = \{\langle \varphi, x\rangle \mid \varphi \text{ is a Boolean formula and } \varphi(x) = 1\}$$

Prove that $\mathrm{FVAL} \in \mathsf{L}$.

6. **[Cycle in directed and undirected graphs]** (10)

Define $\mathrm{CYCLE} = \{\langle G\rangle \mid G \text{ is an directed graph that contains a directed cycle}\}$. Show that CYCLE is NL-complete.

[Hint: The notion of a layered graph might be useful. Given a graph $G = (V, E)$, the k-layering of G, denoted by $G^{(k)}$ is defined as follows. $V(G^{(k)}) = V \times [k]$ (ie., vertices of $G^{(k)}$ are k copies of the vertices of G). For all $i \in 1, \ldots, k-1$ $((u,i),(v,i+1)) \in E(G^{(k)})$ if $(u,v) \in E(G)$, i.e., edges in G become edges in the layered graph across layers.]