Lecture 6 :- Strong duality

Theorem :- (Separating Hyperplane theorem) Let $K \subseteq \mathbb{R}^n$ is a closed convex set. Suppose $p \in \mathbb{R}^n$ is a point s.t. $p \notin K$. Then there is a hyperplane H that (strictly) separates p from K. That is there exist a unit vector $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ s.t. $\forall z \in K$ we have

$$u^T p > \lambda > u^T z .$$

Pf (Sketch). (1) Show that $y = \arg\min_{z \in K} \|p - z\|_2^2$ exists.

(2) Show that the segment $[y, p]$ only intersects the body at y.

(3) Take the segment as the direction of the normal to the hyperplane, and make the plane pass through, e.g., the mid-point of $[y, p]$.

(4) Show that this hyperplane works. ∎

(Homework).

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Convex cones :- A convex cone $C \subseteq \mathbb{R}^n$ is a set s.t. if $a, b \in C$ then $\forall \lambda, \mu \geq 0$, $\lambda a + \mu b \in C$.

[Note that $0 \in C$.]

'Dual' of a set : dual(S) :-

$\mathrm{dual}(S) = \{ y \in \mathbb{R}^n \mid y^T p \leq 0 \quad \forall p \in S \}$.

($\forall S \subseteq \mathbb{R}^n$, $\mathrm{dual}(S)$ is a convex cone.)

Thm:- If C is a closed convex cone then
$$\mathrm{dual}(\mathrm{dual}(C)) = C.$$

Proof:- $D := \mathrm{dual}(C)$. D is a convex cone.

("Weak duality") **Claim**:- $\mathrm{dual}(D) \supseteq C$.

$\mathrm{dual}(D) = \{ y \mid y^T p \leq 0 \quad \forall p \in D \}$

$D = \{ p \mid c^T p \leq 0 \quad \forall c \in C \}$.

Thus $\forall c \in C$ and $p \in D$, we have $c^T p \leq 0$

$\Rightarrow c \in \mathrm{dual}(D)$. Thus $C \subseteq \mathrm{dual}(D)$ ∎

("Strong duality") **Claim**:- $C \supseteq \mathrm{dual}(D)$. Suppose not. Then $\exists p \in \mathrm{dual}(D)$ s.t. $p \notin C$.

But C is a closed convex set. So, by the separating hyperplane theorem, there is a unit vector u and $\lambda \in \mathbb{R}$ s.t.

$$u^T p > \lambda > u^T c \quad \forall c \in C. \quad (*)$$

So for $c = 0$ (as $0 \in C$) we get $\lambda > 0$. — (1)

Question: Can there be a $c \in C$ s.t. $u^T c > 0$?

No! Because since C is a cone, by definition, $\alpha c \in C \quad \forall \alpha \geq 0$. So if

$u^T c > 0$ then $c' := \left(\frac{\lambda}{u^T c}\right) c \in C$, and

$u^T c' = \lambda$ which contradicts (*) !

This is a generally obv. if C is a closed convex cone and $p \notin C$. Then: $\exists u \in dual(C)$ s.t. $u^T p > 0$.

Thus, $u^T c \leq 0 \quad \forall \; c \in C$.

$\Rightarrow \quad u \in dual(C) = D$.

Up till here $p \in dual(D)$ not used.

But $p \in dual(D) \Rightarrow p^T d \leq 0 \quad \forall \; d \in D$.

$\Rightarrow p^T u \leq 0$, but this contradicts (*) again. Thus any $p \in dual(D)$ must also be in C.

— (*).

Farkas lemma:- Let A be a matrix with columns $a_1 \; \ldots \; a_n$. Let C_A be the cone generated by $(a_i)_{i=1}^n$

$$C_A := \left\{ \sum_{i=1}^{n} \lambda_i a_i \;\middle|\; \lambda_i \geq 0 \;\; \forall \; 1 \leq i \leq n \right\}.$$

(Note that C_A is a closed convex cone). Then the following are equivalent :-

(1) There exists x s.t. $Ax = b$, $x \geq 0$

(2) There does not exist y s.t. $A^T y \leq 0$, $b^T y > 0$

Proof:- $\left\{ \exists x : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} \equiv \left\{ b \in C_A \right\}$

(by definition of C_A)

$$\equiv \{ b \in \mathrm{dual}(\mathrm{dual}(C_A)) \}$$

(Using the duality result for the closed convex cone C_A).

$$\equiv \{ b^T y \leq 0 \text{ for all } y \in \mathrm{dual}(C_A) \}.$$

(Note that $\{ y \in \mathrm{dual}\, C_A \} \equiv \{ y^T a_i \leq 0 \quad 1 \leq i \leq n \}$ since the elements of C_A are exactly the non-negative linear combinations of $(a_i)_{i=1}^n$.)

$$\equiv \{ b^T y \leq 0 \text{ for all } y \text{ s.t. } A^T y \leq 0 \}$$

$$\equiv \left\{ \begin{array}{c} \text{There does not exist } y \text{ satisfying} \\ A^T y \leq 0 \\ b^T y > 0 \end{array} \right\}$$

<u>Equivalent Formulation</u>:- <u>Exactly one</u> of the following two programs is feasible.

① $Ax = b$, $x \geq 0$ ② $A^T y \leq 0$, $b^T y > 0$

(<u>Note</u>:- 'Weak duality' version of Farkas lemma says that <u>at most</u> one of ① and ② is feasible, and it is easy to see. The separating hyperplane theorem was need for the "<u>exactly one</u>" claim.)

Back to LP :-

Suppose that PRIMAL is feasible and has finite optimum p^*

$$p^* = \min c^T x$$
$$Ax = b$$
$$x \geq 0.$$

This is equivalent to saying that the set of constraints P_ε given by

$$c^T x = p^* - \varepsilon$$
$$Ax = b$$
$$x \geq 0$$

is feasible for $\underline{\varepsilon = 0}$ and infeasible for $\forall \varepsilon > 0$.

We apply Farkas lemma for even P_ε and see that the constraints D_ε given by.

$$A^T y + z \cdot c \leq 0$$
$$b^T y + z \cdot (p^* - \varepsilon) > 0$$

is infeasible for $\varepsilon = 0$ and feasible for $\forall \varepsilon > 0$.

[This was the crucial part of the argument: $\forall \varepsilon \geq 0$, <u>exactly</u> one of P_ε and D_ε is feasible.]

From this we want to claim that

$$b^T y > p^* - \varepsilon$$
$$A^T y \leq c$$

is feasible for every $\varepsilon > 0$.

(Recall, dual was $d^* = \max b^T y$
$$A^T y \leq c$$
.

We already say $d^* \leq p^*$. So the above claim would show $d^* \geq p^* - \varepsilon$ $\forall \varepsilon > 0$, so that $d^* = p^*$).

Proof of claim: Fix $\varepsilon > 0$. We know $\exists\, y_\varepsilon, z_\varepsilon$ s.t.

$$\left.\begin{aligned} A^T y_\varepsilon + z_\varepsilon \cdot c &\leq 0 \\ b^T y + z_\varepsilon (p^* - \varepsilon) &> 0. \end{aligned}\right\} \quad (1)$$

If $z_\varepsilon < 0$ we are done, because then we get from (1)

$$A^T \left(\frac{y_\varepsilon}{-z_\varepsilon} \right) \leq c.$$

and $$b^T \left(\frac{y_\varepsilon}{-z_\varepsilon} \right) > (p^* - \varepsilon),$$

so that $\dfrac{y_\varepsilon}{-z_\varepsilon}$ is a solution certifying $d^* > p^* - \varepsilon$.

Suppose if possible that $z_\varepsilon \geq 0$.

$(*)$ $b^T y_\varepsilon + z_\varepsilon p^* \geq b^T y_\varepsilon + z_\varepsilon (p^* - \varepsilon) > 0$. (from ①)

So, $(y_\varepsilon, z_\varepsilon)$ satisfy

$$A^T y_\varepsilon + z_\varepsilon \cdot c \leq 0$$

$$b^T y_\varepsilon + z_\varepsilon p^* > 0 \quad (\text{from } (*))$$

But this says that D_ε is feasible at $\varepsilon = 0$! (with feasible solution $(y_\varepsilon, z_\varepsilon)$).

But we know that D_0 is not feasible!

So $z_\varepsilon \geq 0$ is not possible. Hence we are done: ① If p^* is finite then $p^* = d^*$.

From previous work:

② If the primal is unbounded below, then the dual must be infeasible.

③ If the dual is unbounded above, then the primal must be infeasible.

④ If d^* is finite then also $p^* = d^*$ (essentially the same proof as above)

⑤ Both primal and dual infeasible.