Today

Spectral Methods (Part II)

- Adjacency Matrix
  - Wilf's Theorem
  - Hoffman Bd
- Laplacian
  - Drawing Graphs

CSS.205.1

Toolkit in TCS

– Lecture #21

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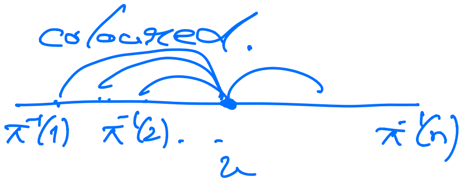
Spectrum of the Adjacency Matrix

Graph Colouring:

Greedy Algorithm:

1. Order the vertices according to some perm π
2. Colour each vertex (in order acc to π) w/ a colour different from its neighbours that have already been coloured.



$$\chi(G) \leq d_{max} + 1$$

(chromatic # graph)

π – number k_π s.t

$$k_\pi = \max_u \#\{v \in N(u) \mid \pi(v) < \pi(u)\}$$

- Greedy: $\boxed{\chi(G) \le k_\pi + 1}$ (if perm π is chosen)

$$\chi(G) \le \min_\pi (k_\pi) + 1$$

Will show that there is a permutation π s.t. $k_\pi \le \mu_1$ . . . (*)

Theorem: [Wilf's Theorem]
$\chi(G) \le \lfloor \mu_1 \rfloor + 1$ (Improvement over d_{max} bd)

Pf: Suffices to show (*).

Last time: $\mu_1 \ge \max_{\emptyset \ne S \subseteq V} d_{ave}(G(S))$

Order vertices in reverse as follows

$\mu_1 \ge d_{ave}$, $\Rightarrow \exists$ vertex v. $\deg(v) \le \mu_1$

π: ———————— v_{n-2} v_{n-1} v_n

Now, for remaining vertices.

$G \setminus \{v_n\}$. $\mu_1 \ge d_{ave}(G \setminus \{v_n\})$

$\exists\, v_{n-1}$ s.t. $\deg_{G \setminus \{v_n\}}(v_{n-1}) \le \mu_1$.

This π satisfies. $k_\pi \le \mu_1$ $\boxtimes$

Hoffman Bound.

Thm: $\chi(G) \ge 1 - \frac{\mu_1}{\mu_n} = \frac{\mu_1 - \mu_n}{-\mu_n}$

Today, we will prove the Hoffman bd for d-regular graphs.

Prove a stronger stmt.

Thm: $\underbrace{\alpha(G)}_{\substack{\text{fractional size} \\ \text{of the largest} \\ \text{independent set}}} \le \frac{-\mu_n}{\mu_1 - \mu_n}$ for ~~d~~-regular graphs

– $\frac{\text{largest ind set size}}{\# \text{ vertices}}$

[Cor: $\chi(G) \ge \frac{\mu_1 - \mu_n}{-\mu_n}$.]

Pf: G – d-regular

$\Rightarrow$ $\mu_1 = d$ & $\varphi_1 = \frac{1}{\sqrt{n}} \mathbb{1}$ ($n = \#$vertices)

Let S be any independent set (say of largest size)

$|S| = \alpha n$.

$f : V \to \mathbb{R}$

Indicator fn of the set S.

$f(v) = \mathbb{1}[v \in S]$

View $f \in \mathbb{R}^V$

– $f = \sum_{i=1}^{n} f_i \varphi_i$ (where φ_i are the orthonormal eigen vectors of A)

– $\langle f, Af \rangle = \sum_{(u,v) \in E} f(u) f(v)$

$= 0$ (f is the indicator of independent set)

$Af = A\left(\sum f_i \varphi_i\right) = \sum \mu_i f_i \varphi_i$

$\langle f, Af \rangle = \left\langle \sum f_i \varphi_i, \sum \mu_i f_i \varphi_i \right\rangle$

$= \sum_{i,j} f_i \mu_j f_j \langle \varphi_i, \varphi_j \rangle$

$= \sum_i \mu_i f_i^2$

$0 = \sum \mu_i f_i^2$

$f_1 = \langle f, \varphi_1 \rangle = \langle f, \frac{1}{\sqrt{n}} \mathbb{1} \rangle = \frac{|S|}{\sqrt{n}} = \alpha\sqrt{n}.$

$\langle f, f \rangle = \sum_{u \in V} f(u) f(u) = |S| = \alpha n.$

$\langle f, f \rangle = \sum f_i^2$

- $f_1 = \alpha\sqrt{n}$
- $\sum f_i^2 = \alpha n$

$$0 = \sum \mu_i f_i^2 = \mu_1 f_1^2 + \sum_{i=2}^{n} \mu_i f_i^2$$

$$\geq \mu_1 \alpha^2 n + \mu_n \sum_{i=2}^{n} f_i^2$$

$$= \mu_1 \alpha^2 n + \mu_n \left(\sum_{i=1}^{n} f_i^2 - f_1^2 \right)$$

$$= \mu_1 \alpha^2 n + \mu_n (\alpha n - \alpha^2 n)$$

Rewriting $\quad \alpha \leq \dfrac{-\mu_n}{\mu_1 - \mu_n}$ $\boxtimes$

The above proof works if we replace adj matrix A w/ any other matrix B s.t

$$B\mathbb{1} = \mathbb{1} \quad \& \quad B(i,j) = 0 \text{ if } (i,j) \notin E$$

(B - has real eigen spectrum)

$$\alpha(G) \leq \vartheta_H(G) \triangleq \min_{\substack{B\text{-symmetric} \\ \lambda_{min}}} \left(\frac{-\lambda_{min}}{1 - \lambda_{min}} \right)$$

$$\begin{cases} B(i,j) = 0 & \text{if } (i,j) \notin E \\ B\mathbb{1} = \mathbb{1} \\ B \succeq \lambda_{min} I \end{cases}$$

Lovasz further strengthened.

$$\alpha(G) \leq \vartheta(G) \triangleq \min_{\substack{\text{sym } C \\ \lambda \in \mathbb{R}}} (\lambda)$$

Lovasz Thetha Function.

$$\begin{cases} C_{ij} = 0 & \forall (i,j) \notin E \\ A + C \preceq \lambda I \end{cases}$$

Laplacian Matrix & Its Spectrum

$L_G = D_G - A_G$.

$$\langle x, L_G x \rangle = \sum_{\{u,v\} \sim E} \left(x(u) - x(v)\right)^2$$

Weighted Graph $\sum_{\{u,v\} \sim E} \omega(u,v) \left(x(u) - x(v)\right)^2$

Graph w/ non-negative, L_G is a PSD

Eigen values $0 = \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n \in \mathbb{R}$

Eigen vectors $\psi_1 \ldots \psi_n \in \mathbb{R}^n$

Least e.value & Corresponding e.vector

$$\lambda_1 = \min_{x \in \mathbb{R}^V} \frac{\langle x, L_G x \rangle}{\langle x, x \rangle}$$

$$= \min_{x} \frac{\sum_{\{u,v\} \sim E} (x(u) - x(v))^2}{\sum_{u \in V} (x(u))^2}$$

$\lambda_1 = 0$ corresponding e.vector $\varphi_1 = \frac{1}{\sqrt{n}} \mathbb{1}$.

If x is an eigenvector of L_G corresponding to evalue 0, then x must be constant on each (connected) component of G.

Multiplicity of 0-eigenvalue = # components of G.

G is connected $\Rightarrow$ $\lambda_2 > 0$.

(λ_2 - measure of how well the graph is connected)

$\hookrightarrow$ Cheeger Inequalities.

Hall's drawing of graphs:

One-dim drawing of graphs:

Goal: Plot the vertices on a straight line such that it minimizes the sum of squared edge distances.

Formulation: Find $x : V \to \mathbb{R}$

s.t $\sum_{\{u,v\} \in E} (x(u) - x(v))^2$ is minimized.

- Obs: (1) Assume
  x is normalized ie
  $\|x\|^2 = \langle x, x \rangle = 1$.

  (2). $x = \text{const}$ minimizes.
  (not a nice picture)
  since all vertices are mapped to same point)

  $x \perp \mathbb{1}$ (x is orthogonal to constant vector).

$$- (3) \quad \min_{\substack{x \in \mathbb{R}^V \\ x \perp \mathbb{1} \\ \langle x, x\rangle = 1}} \sum_{\{u,v\} \sim E} (x(u) - x(v))^2 = \lambda_2$$

& corresponding

x is ψ_2 - second eigenvector.

$\psi_2 : V \to \mathbb{R}$ is "the best" one-dim drawing.

What about 2-dim

Find $x : V \to \mathbb{R}$
$y : V \to \mathbb{R}$ s.t.

$$(*) = \sum_{\{u,v\} \sim E} \left[\begin{pmatrix} x(u) \\ y(u) \end{pmatrix} - \begin{pmatrix} x(v) \\ y(v) \end{pmatrix} \right]^2$$

$$= \sum_{\{u,v\} \sim E} \left[(x(u) - (x(v))^2 + (y(u) - y(v))^2 \right]$$

For the same reasons as before

$\|x\|^2 = \|y\|^2 = 1$, $\langle x, \mathbb{1}\rangle = \langle y, \mathbb{1}\rangle = 0$.

Best soln $x = y = \psi_2$

- degenerate 1-dim drawing.

To prevent, $x \perp y$. $\langle x, y\rangle = 0$

This $\left.\begin{array}{l} x = \psi_2 \\ y = \psi_3 \end{array}\right\}$. as candidate solns.

<u>Thm</u>: $x_1 \ldots x_k : V \to \mathbb{R}$ is a k-dim drawing of the vertices such that

- $\|x_i\|^2 = 1$
- $\langle x_i, x_j \rangle = 0 \quad \forall i \neq j$
- $\langle x_i, \mathbb{1} \rangle = 0 \quad \forall i$

then.

$$\sum_{\{u,v\} \in E} \sum_{i=1}^{k} \left(x_i(u) - x_i(v)\right)^2 \geq \sum_{j=2}^{k+1} \lambda_j$$

Spectra of Toy Graphs.

1. Complete Graph K_n

$$A_{K_n}(i,j) = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{o.w.} \end{cases}$$

L_{K_n} = (Laplacian). $\quad L_{K_n}(i,j) = \begin{cases} -1 & \text{if } i \neq j \\ n-1 & \text{if } i = j \end{cases}$

$$L_{K_n} \mathbb{1} = \bar{0} \quad \left(\lambda_1 = 0 \,;\; \psi_1 = \frac{1}{\sqrt{n}} \mathbb{1} \right)$$

$\varphi \perp \mathbb{1}$

$$(L_{K_n}\varphi)(v) = (n-1)\varphi(v) - \sum_{u \neq v} \varphi(u)$$

$$= (n-1)\varphi(v) + \varphi(v)$$

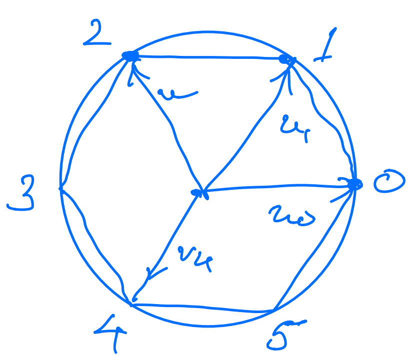
$$= n\,\varphi(v).$$

Hence, any $\varphi \perp \mathbb{1}$, is an e-vector w/ e-value n.

L_{K_n} : Spectrum: e-value $\begin{cases} 0 & \text{w/ mult } 1 \\ n & \text{w/ mult } n-1 \end{cases}$

② Cycle (C_n) / Ring Graph.

$$A_{C_n}(i,j) = \begin{cases} 1 & \text{if } |i-j| = 1 \pmod n \\ 0 & \text{otherwise.} \end{cases}$$



$u_0 \;..\; u_n$

$$u_i = \left(\cos\frac{2\pi i}{n},\; \sin\frac{2\pi i}{n}\right)$$

$$u_{i-1} + u_{i+1} = 2\cos\frac{2\pi}{n}\, u_i.$$

$$\boxed{\begin{aligned} L_G &= dI - A_G \\ A_G\varphi &= \lambda\varphi \\ L_G\varphi &= (d-\lambda)\varphi \end{aligned}}$$

$x(i) = \cos \frac{2\pi i}{n}$; $i = 0, \ldots n-1$

$y(i) = \sin \frac{2\pi i}{n}$; $i = 0, \ldots n-1$

x, y are e-vectors of A_{C_n} w/ e-value $2\cos \frac{2\pi}{n}$.

$x_k(i) = \cos \frac{2\pi k i}{n}$; $i = 0, \ldots n-1$

$y_k(i) = \sin \frac{2\pi k i}{n}$; $i = 0, \ldots n-1$

x_k & y_k are e-vectors of A_{C_n} w/ e-value $2\cos \frac{2\pi k}{n}$.

Eigen-spectrum $1 \leq k \leq n/2$