Today

Spectral Methods (Part III)

- Cayley Graphs & Spectrum
- Random Walk Matrix

CS5.205.1
Toolkit in TCS
- Lecture #22
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Cayley Graphs:

G - group (finite set of elements w/ a binary operation

$G = (S, \cdot)$

$\cdot : G \times G \to G$

$a\, b \mapsto a \cdot b$

(1) $\cdot$ has an identity elt

$\exists\, e \in G, \quad e \cdot a = a \cdot e = a \quad \forall a \in G$

(2) Every elt has an inverse

ie, $\forall a \in G, \exists\, b \in G, \quad ab = ba = e$

(3) Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

If furthermore $\cdot$ is commutative, then G is an Abelian group eg: (1) $G = \mathbb{Z}/p\mathbb{Z}$, $\cdot = +$

(2) $G = \{0,1\}^n$; $\cdot = +$ xor

$(x_1 \ldots x_n) + (y_1 \ldots y_n)$

$= (x_1+y_1, \ldots \quad x_n+y_n)$

↳ xor operation.

Non-Abelian Groups.

(3) $GL_n(\mathbb{R})$ – set of non-singular $n \times n$ matrices w/ real entries

$\cdot$ – product

infinite

(4) $GL_n(\mathbb{Z}/p\mathbb{Z})$

$S \subseteq G$ – set of generators for G if every element in G can be obtained by taking elts in S and applying the group operⁿ

eg: (1) $G = (\{0,1\}^n, +)$

$S = \{e_1, \ldots \quad e_n\}$ $\quad e_i = (0 \ldots 1 0 0 \ldots)$

i^{th} location

(2) $G = (\mathbb{Z}/n\mathbb{Z}, +)$

n - natural number.

$S = \{1\}$

S is closed under inverse if

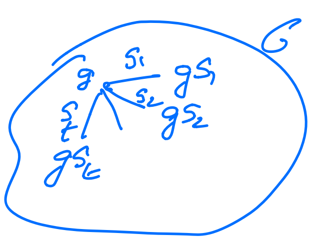
$\forall s \in S \Rightarrow s^{-1} \in S$.

Cycle	Boolean hypercube
C_n - n cycle	H_n
$V = \mathbb{Z}/n\mathbb{Z}$	$V = \{0,1\}^n$
$(a,b) \in E$ if $a = b+1$ or $b = a+1$	$(a,b) \in E$ if $a = b + e_i$ for some $i \in [n]$

Cayley Graph : G - group ; S - set of generators of G. (is closed under inverse)

$Cay(G, S)$: $V = G$

$E = \{(g, gs) \mid g \in G,\ s \in S\}$



Since S is closed under inverse $(g, gs) \in E \Rightarrow (gs, g) \in E$

Cay(G, S) — undirected graph

$$C_n = \mathrm{Cay}((\mathbb{Z}/n\mathbb{Z}, +), \{1, -1\})$$

$$H_n = \mathrm{Cay}((\{0,1\}^n, +), \{e_1 \ldots e_n\})$$

Heisenberg Group:

$$H_3 = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \middle| \; x, y, z \in \mathbb{Z} \right\}$$

Group Operation $= +$

$$T = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$S = T \cup T^{-1}$$

Cayley Graphs over Abelian Groups

Character:

$\chi : G \to \mathbb{C}$ (complexes)

$$\chi(g_1 g_2) = \chi(g_1) \cdot \chi(g_2)$$

eg: $G = (\mathbb{Z}/p\mathbb{Z}, +)$; $\chi_k : G \to \mathbb{C}$

p - prime $\qquad a \to e^{\frac{2\pi a k}{n}}$

for each $k \in \mathbb{Z}/n\mathbb{Z}$

$G = (\{0,1\}^n, +)$; $\chi_S : G \to \mathbb{C}$

$(x_1 \dots x_n) \mapsto \prod_{i \in S} (-1)^{x_i}$

$S \subseteq [n]$ (parity fns)

$\mathcal{F} = \{ f : G \to \mathbb{C} \}$

$\mathcal{F}$ is $|G|$-dimensional vector space.

Consider the inner product on $\mathcal{F}$

$$\langle f, h \rangle = \sum_{g \in G} \overline{f(g)}\, h(g).$$

Proposition:

(1) χ, χ' - character

$\Rightarrow$ $\chi \cdot \chi'$ - character

$\Rightarrow$ χ^{-1} - character

(2) $\left. \begin{array}{l} \chi_0 : G \to \mathbb{C} \\ g \mapsto 1 \end{array} \right\}$ trivial character.

Set of characters form a group.

(3) $$\sum_{g \in G} \chi(g) = \begin{cases} |G|, & \chi\text{- trivial} \\ 0, & \chi\text{- non-trivial} \end{cases}$$

Pf: $\sum_{g \in G} \chi(g)$ (χ is not trivial)

$$= \sum_{g\in G} \chi(gg') \qquad \exists\, g'\in G,\ \chi(g')\neq 1$$

$$= \sum_{g\in G} \chi(g)\chi(g')$$

$$= \chi(g')\sum_{g\in G}\chi(g)$$

(4) $$\langle \chi, \chi' \rangle = \begin{cases} |G| & \text{if } \chi = \chi' \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \chi, \chi' \rangle = \sum_g \overline{\chi(g)}\, \chi'(g)$$

$$= \sum \chi^{-1}(g)\, \chi'(g)$$

$$= \sum (\chi^{-1}\chi')(g)$$

$$= \begin{cases} |G| & \text{if } \chi^{-1}\chi' = \text{trivial} \\ 0 & \text{otherwise} \end{cases}$$

Characters are orthogonal

Abelian Group.

Group of characters $\cong$ Group of

For Abelian groups

Characters form an orthonormal basis for $\mathcal{F}$.

Back to Cayley Graph

$H = Cay(G, S)$. S - set of generators of G closed under inverses.

A_H - adjacency matrix

χ - character.

$$(A_H \chi)(g) = \sum_{s \in S} \chi(gs) = \sum_{s \in S} \chi(g)\chi(s)$$

$$= \chi(g) \underbrace{\left(\sum_{s \in S} \chi(s)\right)}_{\text{independent of } G}$$

Hence χ is an eigenvector of G.
w/ eigenvalue $\sum_{s \in S} \chi(s)$.

- For any Cayley group on G
(irrespective of what S)
the set of evectors = set of characters

- χ is a e.vec of $\begin{cases} A_H \text{ w/ eval } \sum_{s \in S} \chi(s) \\ L_H \text{ w/ eval } |S| - \sum_{s \in S} \chi(s) \end{cases}$

Random Walk Matrix

G - undirected (possibly weighted) graph.

- non-negative weights.

Random Walk: From a vertex u go to vertex v w/ prob proportional to $A(u,v)$

$$\mathrm{Prob}[u \to v] = \frac{A(u,v)}{\sum_{v' \in V} A(u,v')}$$

- Random Walk Matrix

$$D_G^{-1} A_G (u,v) = \frac{A_G(u,v)}{\deg(u)} \quad \Big\} \text{ on weighted graphs}$$

$$= \Pr[u \to v]$$

$$W(u,v) = \Pr[\cancel{\ } \, u \to v]$$

For unweighted graphs: $W_G = D_G^{-1} A_G$

Even for weighted graphs

$$\deg(v) = \sum_{u \in V} A(v,u)$$

$$\Pr[u \to v] = \frac{A_G(u,v)}{\deg(u)}$$

$$D_G = \mathrm{Diag}(\deg(v))$$

Random Walk Matrix (arising from a weighted undirected graph).

$$W_G = D_G^{-1} A_G.$$

Obs: A general random walk matrix.

$$\left. \begin{array}{ll} W \in \mathbb{R}^{n \times n} & \forall u,v. \quad W(u,v) \geq 0 \\ \forall u & \sum_{v \in V} W(u,v) = 1 \end{array} \right\}$$

Not all r.w matrices come from weighted undirected graphs.

Right Multiplication

$W \in \mathbb{R}^{V \times V}$ - r.w matrix

$f : V \to \mathbb{R}$; $f \in \mathbb{R}^V$

Wf - Right multiplication by f

$$(Wf)(u) = \sum_{v \in V} W(u,v) f(v)$$

Right multiplication - corresponds to averaging acc to random walk

$W\mathbb{1} = \mathbb{1}$

ie, $\mathbb{1}$ is a right eigenvector w/ eigenvalue 1.

Left Multiplication:

$p : V \to \mathbb{R}$

$$(pW)(v) = \sum_{u \in V} p(u) W(u,v)$$

Suppose p is a prob. dist on vertices.

$(pW)(v) =$ Prob[RW lands on v when started acc to p]

$$p \underset{1\text{st step}}{\longmapsto} pW \underset{2^{nd}\text{ step}}{\longrightarrow} pW^2 \longrightarrow pW^3$$

— Suppose there exists a prob dist π

ie, $\pi = \pi W$

π - stationary distribution
- left eigenvector of W w/ e-value 1

— For weighted graphs,

$\pi \sim \deg$.

$\rightarrow$ stationary

$$\pi(v) = \frac{\deg(v)}{\sum_{u \in V} \deg(u)} = \frac{\deg}{\Delta}$$

Δ - Total degree

$$\deg W = \deg$$

— Eigen Spectrum of Random Walk Matrix:

π-Inner Product

$$\langle f, g \rangle_\pi = \sum_{v \in V} \pi(v) f(v) g(v)$$

When is W self-adjoint under $\langle , \rangle_\pi$

What is adjoint W^*?

$$\langle f, Wg\rangle = \langle W^* f, g\rangle, \quad \forall f, g \in \mathbb{R}^V$$

$f = \mathbb{1}(v_1)$; $g = \mathbb{1}(v_2)$

$$\langle f, Wg\rangle = \sum_v \pi(v) f(v) (Wg)(v)$$

$$= \pi(v_1) Wg(v_1)$$

$$= \pi(v_1) W(v_1, v_2)$$

$$\langle W^* f, g\rangle = \pi(v_2) (W^* f)(v_2)$$

$$= \pi(v_2) W^*(v_2, v_1)$$

Hence, W^* is the matrix

$$\underbrace{W^*(v_2, v_1) = \frac{\pi(v_1) W(v_1, v_2)}{\pi(v_2)}}_{\text{random walk matrix corresp to time-reversal}}$$

random walk matrix corresp to time-reversal:

$$p \longmapsto pW$$

$p = pWW^* \longleftarrow pW$

If we want W to be self-adjoint

$W^* = W$.

$$W(v_2, v_1) = \frac{\pi(v_1)\, W(v_1, v_2)}{\pi(v_2)} \quad \forall\, v_1, v_2$$

Nicer: $\pi(v_2)\, W(v_2, v_1) = \pi(v_1)\, W(v_1, v_2)$

$\hookrightarrow$ Detailed Balance condition

$\rightarrow$ Reversible Random Walk

Random Walks arising from weighted (undirected) graphs.

$$\pi(v_1) = \frac{\deg(v)}{\Delta} \qquad W(v_1, v_2) = \frac{A(v_1, v_2)}{\deg(v_1)}$$

$$\pi(v_1)\, W(v_1, v_2) = \underbrace{\frac{A(v_1, v_2)}{\Delta}}_{\text{undirected}} = \pi(v_2)\, W(v_2, v_1)$$

Spectral Theorem guarantees an orthonormal eigen decomposition for any self-adjoint RW matrices (under π-dist).

RW matrices arising from weighted undirected graphs are self-adjoint (reversible) (detailed balance).

Eigen decomposition

$v_1, \ldots, v_n \in \mathbb{R}^n$ w/ eigen values $\omega_1, \ldots, \omega_n \in \mathbb{R}$

Obs: (1) $\mathbb{1}$ w/ e.value 1 is a right e.vector.

(2) All e.values have absolute value ≤ 1

(3) $\mathbb{1}$ is the right eigenvector corresponding to 1 w/ largest e.value

Right Eigen space of W

$$\mathbb{1} = v_1, \quad v_2 \ldots \qquad v_n$$

$$1 = \omega_1 \geq \omega_2 \ldots \qquad \geq \omega_n \geq -1$$

Left Eigen Space of W

— Next lecture.