Today

Spectral Methods (Part IV)

- Random Walk Matrix
- Convergence
- Expander Mixing Lemma
- Cheeger Inequalities

CSS.205.1
Toolkit in TCS
- Lecture #23
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Last time:

Unweighted graph : $W_G = D_G^{-1} A_G$

$\downarrow$

Weighted Graphs $\qquad W_G = D_G^{-1} A_G$
(non-negative weights)
(w/ no isolated vertices) $\qquad \deg(v) = \sum_{u \in V} w(v,u)$

$W \mathbb{1} = \mathbb{1}$
$W(u,v) \geq 0, \ \forall u,v$

} Requirements for any random walk matrix.

Stationary Dist:

$\pi : V \to \mathbb{R}$.

$\sum \pi(v) = 1$ $\qquad \pi(v) \sim \deg(v)$

$\pi(v) = \deg(v) \Big/ \sum_{u \in V} \deg(u)$ $\qquad \pi^T W = \pi^T$

Inner Product: $\langle f, g \rangle_\pi = \sum_{v \in V} \pi(v) f(v) g(v)$

Reversible Random Walk:

$$\pi(u) W(u,v) = \pi(v) W(v,u) \quad \forall u, v \in V$$

$W = W^*$ (under $\langle , \rangle_\pi$ inner product)

Eigen values of W:

$1 = \omega_1 \geq \omega_2 \geq \quad \omega_n \geq -1 \quad \in \mathbb{R}$

Right Eigenvectors:

$\mathbb{1} = v_1 \quad v_2 \, . \quad v_n \quad \in \mathbb{R}^n$

(orthonormal)

Left Eigenvectors:

What are the vectors $v_1', \ldots v_n'$

s.t $\quad v_i'^T W = \omega_i v_i^T$ or

$$W^T v_i' = \omega_i v_i'$$

W is self-adjoint

$\langle f, Wg \rangle = \langle Wf, g \rangle \, , \, \forall f, g : V \to \mathbb{R}.$

$$\langle f, g\rangle = \sum_{v\in V} \pi(v) f(v) g(v) = f^T \Pi g.$$

$$\Pi = \mathrm{diag}(\pi)$$

$$\langle Wf, g\rangle = (Wf)^T \Pi g = f^T W^T \Pi g$$

$$\langle f, Wg\rangle = f^T \Pi W g$$

Reversibility $\Rightarrow$ $f^T \Pi W g = f^T W^T \Pi g \quad \forall f, g$

$$\Pi W = W^T \Pi$$

ie, $W^T = \Pi W \Pi^{-1}$

For $i \in V$ $\quad W v_i = \omega_i v_i$

$$v_i' = \Pi v_i$$

$$W^T v_i' = (\Pi W \Pi^{-1})(\Pi v_i) = \Pi W v_i = \omega_i \Pi v_i$$
$$= \omega_i v_i'$$

Left Eigenvectors

$$\pi = \Pi \mathbb{1}, \quad \Pi v_2, \ldots \quad \Pi v_n$$

w/ evalues

$$1 = \omega_1 \geq \omega_2 \ldots \quad \geq \omega_n \geq -1$$

$$\Pi W = W^T \Pi \quad ; \quad (\Pi W)^T = W^T \Pi^T$$
$$= W^T \Pi = \Pi W$$

ΠW - symmetric matrix.

$W = \Pi^{-1}(\Pi W)$ (recall $W_G = D_G^{-1} A_G$)

Reversible Random Walk : W
Weighted graph - underlying W } equivalent

$A = \Pi W$

Observations:

① $1 = \omega_1 \geq \omega_2$: dim (eigenspace corresponding to 1. e.v)

= # components

Underlying graph is connected
$1 = \omega_1 > \omega_2$.

② $\omega_n \geq -1$

Underlygraph is bipartite $\Leftrightarrow \omega_n = -1$
$\Leftarrow$ (Pset 4).

Random walk mix

Bottlenecks for mixing:

(1) Underlying graph - disconnected (ie, $\omega_2 = 1$)

(2) " " - bipartite. (ie, $\omega_n = -1$)

$$\omega = \max\{\omega_2, -\omega_n\}$$

$\omega < 1$ "$\Rightarrow$" Random-walk converges.

(Removing bottlenecks)

<u>Setup</u>:

p_0 - initial distribution on vertices (possibly supported on only one vertex).

$t \leftarrow 1$ to ∞.

$$p_t^T \leftarrow p_{t-1}^T \cdot W$$

<u>Qn</u>: Does p_t converge to π?

Total-variation distance between p_t & π

$$= d_{TV}(p_t, \pi).$$

$$= \frac{1}{2} \|p_t - \pi\|_1$$

$$= \frac{1}{2} \sum_{v \in V} |P_t(v) - \pi(v)|$$

$$= \frac{1}{2} \sum_{v \in V} \pi(v) \left| \frac{P_t(v)}{\pi(v)} - 1 \right|$$

$$= \frac{1}{2} \sum_{v \in V} \pi(v) \left| (\Pi^{-1} P_t)(v) - 1 \right|$$

$$\triangleq \frac{1}{2} \| \Pi^{-1} P_t - \mathbb{1} \|_{1,\pi}$$

$$\|\varphi\|_{k,\pi} = \sqrt[k]{\sum_{v \in V} \pi(v) \cdot \varphi(v)^k}$$

$$\leq \frac{1}{2} \| \Pi^{-1} P_t - \mathbb{1} \|_{2,\pi}$$

$\| \Pi^{-1} P_t - \mathbb{1} \|_{2,\pi}$ - how does this quantity change with each step of the random walk.

$$\| \Pi^{-1} P_t - \mathbb{1} \|_{2,\pi} \quad \text{vs} \quad \| \Pi^{-1} P_{t+1} - \mathbb{1} \|_{2,\pi}$$

$$\| \Pi^{-1} (W^T P_t) - \mathbb{1} \|_{2,\pi}$$

Write prob. vector P_t in its (left) eigen basis

$$P_t = \sum_{i=1}^{n} \alpha_i v_i' \qquad \Big(v_i' = \Pi v_i$$

- left eigenvectors of W

$$= \alpha_1 v_1' + \sum_{i=2}^{n} \alpha_i v_i'$$

v_i - right eigenvectors)

$$= \alpha_1 \pi + \sum_{i=2}^{n} \alpha_i v_i'$$

$$= \alpha_1 \pi + \sum_{i=2}^{n} \alpha_i \Pi v_i$$

Observe that $\alpha_1 = 1$ since P_t satisfies $\sum_{v \in V} P_t(v) = 1$

$$(\text{ie } \sum_{u \in V} \Pi v_i(u)$$

$$= \sum_{u \in V} \pi(u)\, v_i(u).$$

$$= \langle v_i, \mathbb{1} \rangle_\pi = 0.$$

$$P_t = \pi + \sum_{i=2}^{n} \alpha_i \Pi v_i$$

$$P_{t+1} = W^T P_t = W^T\left(\pi + \sum_{i=2}^{n} \alpha_i \Pi v_i\right)$$

$$= \pi + \sum_{i=2}^{n} \alpha_i W^T \Pi v_i$$

$$P_{t+1} = \pi + \sum_{i=2}^{n} \alpha_i \omega_i \Pi v_i$$

$$\Pi^{-1} P_t - \mathbb{1} = \Pi^{-1}\left(\pi + \sum_{i=2}^{n} \alpha_i \Pi v_i\right) - \mathbb{1}$$

$$= \mathbb{1} + \sum_{i=2}^{n} \alpha_i v_i - \mathbb{1}$$

$$= \sum_{i=2}^{n} \alpha_i v_i$$

Similarly $\Pi^{-1} P_{t+1} - \mathbb{1} = \sum_{i=2}^{n} \alpha_i \omega_i v_i$

$$\| \Pi^{-1} P_t - \mathbb{1} \|_{2,\pi}^2 = \| \sum_{i=2}^{n} \alpha_i v_i \|_{2,\pi}^2$$

$$= \sum_{i=2}^{n} \alpha_i^2$$

$$\| \Pi^{-1} P_{t+1} - \mathbb{1} \|_{2,\pi}^2 = \sum_{i=2}^{n} \omega_i^2 \alpha_i^2$$

$$\leq \omega^2 \sum_{i=2}^{n} \alpha_i^2 \quad (\text{since } \omega = \max\{\omega_2, -\omega_n\})$$

$$d_{TV}(P_t, \pi) \leq \frac{1}{2} \| \Pi^{-1} P_t - \mathbb{1} \|_{2,\pi}$$

$$\leq \frac{\omega^t}{2} \| \Pi^{-1} P_0 - \mathbb{1} \|_{2,\pi} \leq \frac{\omega^t}{2} \frac{1}{\sqrt{\pi_{min}}}$$

For any start distribution P_0

$$\| \Pi^{-1} P_0 - \mathbb{1} \|_{2,\pi} \leq \frac{1}{\sqrt{\underbrace{\min_v \pi(v)}_{\pi_{min}}}}$$

Expander Mixing Lemma.

$f, g : V \to \mathbb{R}.$

$$\mu_f = \mathbb{E}_\pi [f(v)] = \sum_{v \in V} \pi(v) f(v)$$

Similarly μ_g.

$$\langle f, Wg\rangle_\pi = \sum_{(v,u)\in V\times V} \pi(v)\, W(v,u)\, f(v)\, g(u)$$

$$= \mathop{\mathbb{E}}_{\substack{v\sim\pi \\ u\sim RW(v)}}[f(v)g(u)]$$

$$\mathop{\mathbb{E}}_{(u,v)\sim E}[f,g] \quad \text{vs} \quad \mathop{\mathbb{E}}_{\pi}[f]\cdot\mathop{\mathbb{E}}_{\pi}[g]$$

$$\| $$

$$\mathop{\mathbb{E}}_{\substack{v\sim\pi \\ u\sim\pi}}[f(u)g(v)]$$

Expander Mixing Lemma.

$$\left|\mathop{\mathbb{E}}_{\substack{u\sim\pi \\ v\sim RW(u)}}[f(u)g(v)] - \mu_f\mu_g\right| \le \omega\, \sigma_f\, \sigma_g.$$

where $$\sigma_f = \mathop{\mathbb{E}}_{v\sim\pi}[f^2(v)] - \mathop{\mathbb{E}}_{v\sim\pi}[f(v)]^2$$

$$= \mathop{\mathbb{E}}_{\pi}[f^2] - \left(\mathop{\mathbb{E}}_{\pi}[f]\right)^2$$

Pf: Proof is a simple application of writing f & g in their eigen decomposition.

$$f = \sum \alpha_i v_i = \alpha_1 \mathbb{1} + \sum_{i=2}^{n} \alpha_i v_i$$

$$g = \sum \beta_i v_i = \beta_1 \mathbb{1} + \sum_{i=2}^{n} \beta_i v_i$$

$$\mathop{\mathbb{E}}_{\substack{i \sim \pi \\ j \sim RW(i)}}[f(i) g(j)] = \langle f, Wg \rangle$$

$$= \langle \sum \alpha_i v_i , W(\sum \beta_i v_i) \rangle$$

$$= \langle \sum \alpha_i v_i , \sum \omega_i \beta_i v_i \rangle$$

$$= \sum \omega_i \alpha_i \beta_i$$

$$= \alpha_1 \beta_1 + \sum_{i=2}^{n} \omega_i \alpha_i \beta_i$$

$$\alpha_1 = \langle f, \mathbb{1} \rangle = \sum \pi(i) f(i) = \mathop{\mathbb{E}}_{\pi}[f]$$

$$= \mu_f$$

$$\beta_1 = \mu_g .$$

$$|\langle f, Wg \rangle - \mu_f \mu_g| = \left| \sum_{i=2}^{n} \omega_i \alpha_i \beta_i \right|$$

$$\leq \left| \omega \sum_{i=2}^{n} \alpha_i \beta_i \right|$$

$$= \omega \left| \sum_{i=2}^{n} \alpha_i \beta_i \right|$$

$$\leq \omega \sqrt{\sum_{i=2}^{n} \alpha_i^2} \sqrt{\sum_{i=2}^{n} \beta_i^2} = \omega \sigma_f \sigma_g .$$

$$\sum_{i=2}^{n} \alpha_i^2 = \sum_{i=1}^{n} \alpha_i^2 - \alpha_1^2 = \langle f, f \rangle_\pi - \mu_f^2$$

$$= \mathbb{E}_{i \sim \pi}\left[f^2(i)\right] - \mu_f^2$$

$$= \sigma_f^2$$