Today

Spectral Methods (Part V)

– Cheeger Inequalities

CS5.205.1
Toolkit in TCS
– Lecture #24
(12 May '21)
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Recall: W – random walk matrix.

Vertex Set $V = \{1, 2, \ldots n\}$.

$$\begin{cases} W(i,j) \geq 0 \quad \forall\ i,j \in V \\ W\mathbb{1} = \mathbb{1} \end{cases}$$

Generalization of $D^{-1}A$
(A – adjacency matrix)
$D = \text{Diag}(\deg(i))$.

– Stationary Dist $\pi^T W = \pi^T$

$$\begin{cases} \pi : V \to \mathbb{R};\quad \pi(i) \geq 0,\ \forall i \\ \sum \pi(i) = 1 \end{cases}$$

Inner Product: $\langle f, g \rangle_\pi = \sum_{i \in V} \pi(i) f(i) g(i)$

Adjoint: $\langle f, Wg \rangle_\pi = \langle W^* f, g \rangle_\pi$

W^* - random walk corresponding to the time-reversal walk

$$W^*(i,j) = \frac{W(j,i)\,\pi(j)}{\pi(i)}$$

Reversible RW: $W^* = W$ (self-adjoint under $\langle , \rangle_\pi$)

Reversible RW - Spectral Decomposition

e.val: $1 = \omega_1 \geq \omega_2 \ldots \geq \omega_n \geq -1$

e.vec (right) $\mathbb{1} = v_1,\ v_2 \ .\ .\ v_n.$

e.vec (left) $\pi = \Pi v_1\ ;\ \Pi v_2, .\ .\ \Pi v_n$

$$\Pi = \mathrm{Diag}(\pi)$$

$$\Pi W = W \Pi$$

$$W = \Pi^{-1} (\underbrace{\Pi W}_{\text{symmetric}})$$

Similar $D^{-1}A$

Underlygraph: edge weights ΠW

ie, $a(i,j) = \pi(i) W(i,j)$

$$\deg(i) = \sum_j a(i,j) = \sum_j \pi(i) W(i,j)$$
$$= \pi(i)$$

Convergence: $\omega = \max\{\omega_2, -\omega_n\}$.

Expander Mixing Lemma

Laplacian of Random Walk (reversible)

$$\mathcal{L} = I - W$$
(Normalized Laplacian)

For graphs,
$$\mathcal{L} = D - A$$
$$\mathcal{L} = D(I - D^{-1}A)$$

Operator $\mathcal{L}$

Quadratic form corresponding to $\mathcal{L}$.

$$\begin{cases} \mathcal{L} : \mathbb{R}^V \to \mathbb{R}^V \\ \quad f \mapsto \mathcal{L}f \\ (\mathcal{L}f)(i) = \sum_j \mathcal{L}(i,j) f(j) = f(i) - \sum_j W(i,j) f(j) \\ \qquad\qquad = \sum_j W(i,j)(f(i) - f(j)) \end{cases}$$

Quadratic Form:

$$\langle f, \mathcal{L}f \rangle_\pi = \sum_i \pi(i) f(i) (\mathcal{L}f)(i)$$
$$= \sum_i \pi(i) f(i) \sum_j W(i,j)(f(i) - f(j))$$
$$= \sum_{i,j} \pi(i) W(i,j) \; f(i)(f(i) - f(j)) \quad \dots \quad (1)$$

$$\langle f, \mathcal{L}f \rangle_\pi = \frac{\langle f, \mathcal{L}f \rangle_\pi + \langle \mathcal{L}^* f, f \rangle}{2}.$$

$$\langle L^* f, f\rangle = \langle f, L^* f\rangle \qquad\qquad L^* = (I-W)^* = I - W^*$$

$$= \sum_{i,j} \pi(i) W^*(i,j)\, f(i)\big(f(i)-f(j)\big) \qquad\qquad W^*(i,j) \text{ sat } \pi(i) W^*(i,j) = \pi(j) W(j,i)$$

$$= \sum_{i,j} \pi(j) W(j,i) f(i) \big(f(i)-f(j)\big)$$

$$= \sum_{i,j} \pi(i) W(i,j)\, f(j)\big(f(j)-f(i)\big) \quad \ldots (2)$$

$$\langle f, Lf\rangle = \frac{1}{2}\Big(\langle f, Lf\rangle + \langle L^* f, f\rangle\Big)$$

$$= \frac{1}{2}\sum_{i,j} \pi(i) W(i,j) \big(f(i)-f(j)\big)^2$$

$$= \sum_{\{i,j\}} \pi(i) W(i,j) \big(f(i)-f(j)\big)^2$$

↳ unordered pairs

Prop: For any random walk W (not necessarily reversible), the quad form corresponding to the Laplacian

$$\langle f, Lf\rangle = \sum_{\{i,j\}} \pi(i) W(i,j) \big(f(i)-f(j)\big)^2$$

Furthermore, for reversible RW W

$L^* = L$. L is PSD operator under $\langle\ ,\ \rangle_\pi$

Reversible RW W ; $L = I - W$

e.val. $0 = \underset{1-\omega_1}{\underset{\shortparallel}{\lambda_1}} \le \underset{1-\omega_2}{\underset{\shortparallel}{\lambda_2}} \le \underset{1-\omega_3}{\underset{\shortparallel}{\lambda_3}} \dots \le \underset{1-\omega_n}{\underset{\shortparallel}{\lambda_n}} \le 2$

e.vect (right) $v_1 \quad v_2 \quad \dots \quad v_n \in \mathbb{R}^n$

Graph is connected $\iff \omega_2 < 1$

$\iff \lambda_2 > 0$

<u>Cheeger Inequalities:</u>

Quantified version:

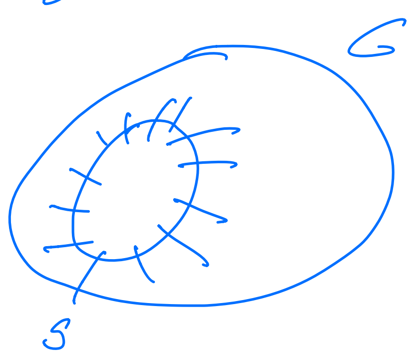
"Graph is connected $\iff \lambda_2 > 0$"

Conductance (Measure of how well the graph is connected).

$S \subseteq V$

Regular graphs

d-regular



$\partial S = \{(i,j) \,/\, i \in S,\ j \notin S\}$

$\varphi(S) = \dfrac{|\partial S|}{d \cdot |S|}$

$(|S| \le |V|/2)$

$$\varphi(G) = \min_{\substack{S: \\ 0 \le |S| \le |V|/2}} \varphi(S)$$

Irregular Unweighted Graphs

$$\partial S = \{(i,j) \mid i \in S, j \notin S\}.$$

$$d(S) = \sum_{i \in S} \deg(i)$$

$$\varphi(S) = \frac{|\partial S|}{d(S)} \quad \text{for } S \text{ s.t. } d(S) \le \frac{d(V)}{2}$$

Weighted Graphs.

$$\varphi(S) = \frac{\partial S}{d(S)}$$

$$\partial S = \sum_{\substack{i \in S \\ j \notin S}} a(i,j)$$

$$d(S) = \sum_{i \in S} \sum_{j \in V} a(i,j)$$

$$\varphi(G) = \min_{\substack{S: \\ 0 < d(S) \le \frac{d(V)}{2}}} \frac{\partial S}{d(S)}$$

G is connected $\iff$ $\lambda_2 > 0$

$\varphi_- > 0 \iff \lambda_2 > 0$

Cheeger Inequalities:

$$\frac{\lambda_2}{2} \le \varphi(G) \le \sqrt{2\lambda_2}$$

Cheeger 70's (manifolds)

Graphs & Random Walks

Jerrum - Sinclair
Alon - Milman
Lawler - Sokal
Varopoulos
Dodziuk

80's.

Polya - Szego - 50's

Today: Fan-Chung / Girish Varma.
presentation.

Easy Direction: $\varphi(G) \geq \lambda_2/2$

Small conductance $\Rightarrow$ small 2^{nd} eigen value.

Lemma: For any set $S \subseteq V$, $\pi(S) \leq 1/2$,
$$\varphi(S) \geq \lambda_2/2.$$

Pf: Prove using the Courant-Fischer characterization.

$$\lambda_2 = \min_{f \perp \mathbb{1}} \frac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi}$$

Let S be a set $S \subseteq V$ s.t. $\pi(S) \leq 1/2$

One natural choice is $\mathbb{1}_S$

$$\langle \mathbb{1}_S, L\mathbb{1}_S \rangle_\pi = \sum_{\{i,j\}} \pi(i)\, W(i,j) \left(\mathbb{1}_S(i) - \mathbb{1}_S(j)\right)^2$$

$$= \partial S$$

$$\langle \mathbb{1}_S, \mathbb{1}_S \rangle_\pi = \sum_{i\in V} \pi(i)\, \mathbb{1}_S(i)\, \mathbb{1}_S(i) = \pi(S)$$

However $\mathbb{1}_S \not\perp \mathbb{1}$

$$f = \mathbb{1}_S - \sigma \mathbb{1}$$

$$0 = \langle f, \mathbb{1} \rangle_\pi = \langle \mathbb{1}_S - \sigma\mathbb{1}, \mathbb{1} \rangle_\pi = \pi(S) - \sigma \cdot 1 = 0$$

(ie, $\sigma = \pi(S)$)

Use $f = \mathbb{1}_S - \pi(S)\mathbb{1}$ satisfies $f \perp \mathbb{1}$

$$\langle f, Lf \rangle_\pi = \langle \mathbb{1}_S, L\mathbb{1}_S \rangle_\pi \quad (\text{since } f = \mathbb{1}_S - \sigma\mathbb{1})$$

$$= \partial S.$$

$$\langle f, f \rangle_\pi = \langle \mathbb{1}_S - \pi(S)\mathbb{1}, \mathbb{1}_S - \pi(S)\mathbb{1} \rangle_\pi$$

$$= \langle \mathbb{1}_S, \mathbb{1}_S \rangle_\pi - 2\pi(S) \langle \mathbb{1}, \mathbb{1}_S \rangle_\pi + \pi^2(S) \langle \mathbb{1}, \mathbb{1} \rangle_\pi$$

$$= \pi(S) - 2\pi(S) \cdot \pi(S) + \pi^2(S) \cdot 1$$

$$= \pi(S) - \pi^2(S)$$

$$= \pi(S)(1 - \pi(S))$$

$$\lambda_2 \le \frac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi} = \frac{\partial S}{\pi(S)(1-\pi(S))} \le \frac{2\partial S}{\pi(S)} = 2\varphi(S).$$

$\boxtimes$

[Conductance (sometimes defined)

$$\varphi(S) = \frac{d(V)\cdot|\partial S|}{d(S)\cdot d(V\setminus S)}$$ / Won't use in our lectures.

Hard Direction: $\varphi(G) \leq \sqrt{2\lambda_2}$

Small eigen-value $\Rightarrow$ Cut w/ small conductance.

Proof will be constructive.

ie, given a $f: V \to \mathbb{R}$ s.t

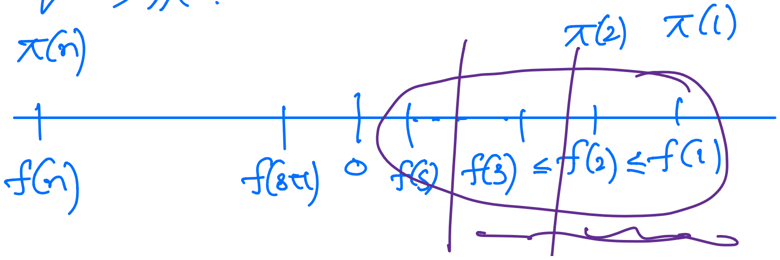
(1) $f \perp \mathbb{1}$ & f is non-zero

(2) $$\frac{\langle f, Lf\rangle_\pi}{\langle f, f\rangle_\pi} = \rho$$

Will construct a set $S \subseteq V$, $\pi(S) \leq 1/2$

s.t. $$\varphi(S) \leq \begin{cases} \sqrt{\rho(2-\rho)} & \text{if } \rho \leq 1 \\ 1 & \text{if } \rho > 1 \end{cases}$$

$f: V \to \mathbb{R}$.



Relabel vertices.

$\pi(\{i \mid f(i) > 0\}) \leq 1/2$

$g : V \to \mathbb{R}$ - proper function

(1) $g(i) \geq 0$, $\forall i \in V$

(2) $\pi(\{i / g(i) > 0\}) \leq 1/2$.

Lemma 1: Given non-zero f st $f \perp \mathbb{1}$

& $\rho = \frac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi}$

there exists a proper g s.t

$\frac{\langle g, Lg \rangle_\pi}{\langle g, g \rangle_\pi} \leq \rho$

Lemma 2: If g is proper then

$\varphi(G) \leq \sqrt{\rho(2-\rho)}$

where $\rho = \frac{\langle g, Lg \rangle_\pi}{\langle g, g \rangle_\pi}$