Today

Spectral Methods (Part VI)

- Cheeger Inequalities
- Expander Graphs

CSS.205.1
Toolkit in TCS
- Lecture #25
(17 May '21)
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Recall (from last time).

Cheeger Inequalities.

$$\frac{\lambda_2}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2}$$

where λ_2 = second e.value of the Laplacian of the (reversible) RW matrix

$$\varphi(G) = \min_{S : 0 < \pi(S) \leq 1/2} \varphi(S)$$ - conductance of a graph.

$$\varphi(S) = \frac{|\partial S|}{\pi(S)}$$

Today: **Hard direction: $\varphi(G) \leq \sqrt{2\lambda_2}$**

Laplacian small 2^{nd} evalue $\Rightarrow$ Cut w/ small conductance

Constructive Proof.

Given a $f: V \to \mathbb{R}$ (the second evector)

s.t $\begin{cases} (1) \ f \perp \mathbb{1} \ \& \ f \text{ is non-zero.} \\ (2) \ \dfrac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi} = \rho \end{cases}$

Will construct a set $S \subseteq V$, $\pi(S) \leq 1/2$.
satisfying.

$$\varphi(S) \leq \begin{cases} \sqrt{\rho(2-\rho)} & \text{if } \rho \leq 1 \\ 1 & \text{if } \rho > 1 \end{cases}$$

$g: V \to \mathbb{R}$ is a proper fn:

(i) $g(i) \geq 0 \quad \forall i \in V$

(ii) $\pi(\{i \mid g(i) > 0\}) < 1/2$

Lemma 1: Given non-zero 2^{nd} evector $f: V \to \mathbb{R}$ s.t

(i) $\langle f, \mathbb{1} \rangle_\pi = 0$

(ii) $\lambda_2 = \dfrac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi}$

there exists a non-zero proper $g: V \to \mathbb{R}$

s.t $\dfrac{\langle g, Lg \rangle_\pi}{\langle g, g \rangle_\pi} \leq \lambda_2$

Pf: Assume $\pi(\{i \mid f(i) \geq 0\}) < 1/2$, if not replace f with $-f$.

Define $g(i) = \begin{cases} f(i) & \text{if } f(i) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

g is proper $\left(\begin{array}{l} g \text{ is non-negative} \\ \pi(\{i \mid g(i) > 0\}) < 1/2 \end{array} \right)$

How is $\frac{\langle g, \mathcal{L} g\rangle_\pi}{\langle g, g\rangle_\pi}$ compared to $\frac{\langle f, \mathcal{L} f\rangle_\pi}{\langle f, f\rangle_\pi}$

$S = \{i \mid g(i) > 0\} = \{i \mid f(i) > 0\}$.

For $i \in S$

$$
\begin{aligned}
(\mathcal{L} g)(i) &= \sum_j w(i,j)\,(g(i) - g(j)) \\
&\leq \sum_j w(i,j)\,(f(i) - f(j)) \\
&= (\mathcal{L} f)(i) = \lambda_2 f(i) = \lambda_2 g(i)
\end{aligned}
$$

$$
\begin{aligned}
\langle g, \mathcal{L} g\rangle_\pi &= \sum_{i \in S} \pi(i)\, g(i)\, (\mathcal{L} g)(i) \\
&\leq \sum_{i \in S} \pi(i)\, g(i)\, \lambda_2\, g(i) \\
&= \lambda_2 \langle g, g\rangle_\pi
\end{aligned}
$$

$\boxtimes$

Lemma 2: If g is proper non-zero fn

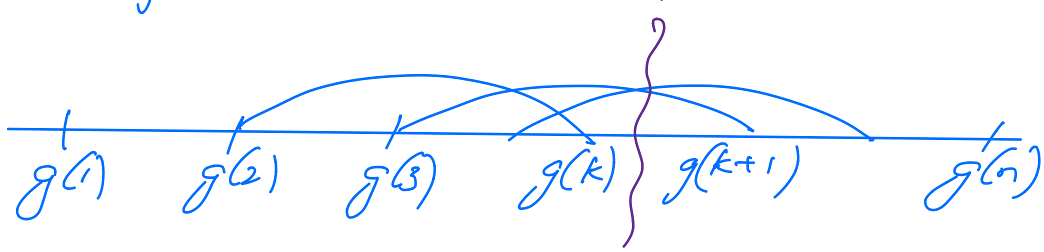
$$\varphi(G) \le \sqrt{\rho(2-\rho)} \quad \text{where } \rho = \frac{\langle g, \mathcal{L}g\rangle_\pi}{\langle g, g\rangle_\pi}$$

Pf: g is proper
Wlog order vertices s.t.

$$\begin{cases} g(1) \ge g(2) \ge \cdots \ge g(s) > 0 = g(s+1) = \cdots = g(n) \\ \pi(\{1,2,\dots s\}) < 1/2 \end{cases}$$

Consider.

$$\sum_{i<j} \pi(i)\, w(i,j)\left(g^2(i) - g^2(j)\right) \quad \cdots (*)$$



$$(*) = \sum_{i<j} \pi(i)\, w(i,j) \sum_{k=i}^{j-1} \left(g^2(k) - g^2(k+1)\right)$$

$$= \sum_{k=1}^{s} \left(g^2(k) - g^2(k+1)\right) \partial(\{1,2 \dots k\})$$

$$\ge \sum_{k=1}^{s} \left(g^2(k) - g^2(k+1)\right) \varphi \cdot \pi(\{1, \dots k\})$$

$$= \varphi \sum_{k=1}^{s} \left(g^2(k) - g^2(k+1)\right) \pi(\{1, \dots, k\})$$

$$= \varphi \sum_{k=1}^{s} g^2(k) \cdot \pi(k)$$

$$= \varphi \langle g, g \rangle_\pi \quad \dots \quad (1)$$

$$(*) = \sum_{i<j} \pi(i) w(i,j) \left(g^2(i) - g^2(j)\right)$$

$$= \sum_{i<j} \left\{ \sqrt{\pi(i) w(i,j)} \left(g(i) - g(j)\right) \right\} \cdot \left\{ \sqrt{\pi(i) w(i,j)} \left(g(i) + g(j)\right) \right\}$$

$$\leq \left(\sum_{i<j} \pi(i) w(i,j) \left(g(i) - g(j)\right)^2 \right)^{1/2} \cdot \left(\sum_{i<j} \pi(i) w(i,j) \left(g(i) + g(j)\right)^2 \right)^{1/2}$$

$$= \left(\langle g, Lg \rangle_\pi \right)^{1/2} \left(\sum_{i<j} \pi(i) w(i,j) \left(2\left(g^2(i) + g^2(j)\right) - \left(g(i) - g(j)\right)^2 \right) \right)^{1/2}$$

$$= \langle g, Lg \rangle_\pi^{1/2} \left(2 \sum_i \pi(i) g^2(i) - \sum_{i<j} \pi(i) w(i,j) \left(g(i) - g(j)\right)^2 \right)^{1/2}$$

$$= \langle g, Lg \rangle_\pi^{1/2} \left(2 \langle g, g \rangle_\pi - \langle g, Lg \rangle_\pi \right)^{1/2}$$

$$(*) \leq \sqrt{\langle g, Lg \rangle_\pi \left(2 \langle g, g \rangle_\pi - \langle g, Lg \rangle_\pi \right)} \quad \dots \quad (2)$$

$$\varphi \langle g, g\rangle_\pi \le \sqrt{\langle g, \mathcal{L} g\rangle_\pi (2\langle g, g\rangle_\pi - \langle g, \mathcal{L} g\rangle_\pi)}$$

ie, $\varphi \le \sqrt{\rho(2-\rho)}$ where $\rho = \dfrac{\langle g, \mathcal{L} g\rangle_\pi}{\langle g, g\rangle_\pi}$

$\boxtimes$

<u>Remark:</u> Proof demonstrated that there is a set $S = \{1, 2, .. i\}$, $i \le s$

$$\varphi(S) \le \sqrt{2\lambda_2}$$

Fiedler's Algorithm:

Input: RW matrix W & a vector f, s.t $\langle f, \mathbb{1}\rangle_\pi$

$\langle f, \mathcal{L} f\rangle_\pi \le \alpha \langle f, f\rangle_\pi$

Output: A cut $(S, V \setminus S)$ s.t

$\pi(S) < 1/2$ & $\varphi(S) \le \sqrt{\alpha(2-\alpha)}$

1. If $\pi(\{i \mid f(i) > 0\}) \ge 1/2$, then $f \leftarrow -f$
2. Order the vertices s.t
   $f(1) \ge f(2) \ldots \ge f(s) > 0 \ge f(s+1) \ge \ldots \ge f(n)$
3. Compute $\varphi(\{1, \ldots i\})$ for $i \leftarrow 1$ to s & o/p the cut w/ least conductance

Tightness of Cheeger Inequalities

$$\frac{\lambda_2}{2} \leq \varphi(G) \leq \sqrt{2\lambda_2}$$

1. $\varphi(G) \geq \lambda_2/2$.

eg: Hypercube $- \mathrm{Cay}(\{0,1\}^n, (e_1 \dots e_n))$

λ_2 : Evectors corresponding to second smallest e-value are the character corresponding to singleton set.

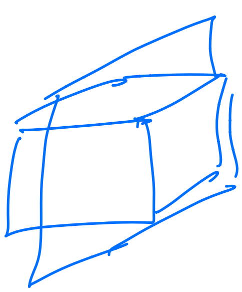
$$\chi_S : V \to \mathbb{R} : \quad \chi_S(x_1 \dots x_n) = (-1)^{\sum_{i \in S} x_i}$$

$$\chi_{\{i\}} : \{0,1\}^n \to \mathbb{R}$$
$$(x_1 \dots x_n) \mapsto (-1)^{x_i}$$

2^{nd} e-value $= 2/n$.

Dictator Cuts : $i \in [n]$

$$S_i = \{x \in \{0,1\}^n \mid x_i = 0\}.$$



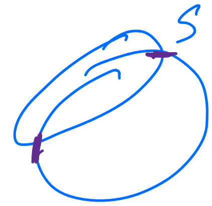
$$\varphi(S_i) = \frac{\partial S_i}{\pi(S_i)} = \frac{1}{n}.$$

2. $\varphi(G) \le \sqrt{2\lambda_2}$:

Cycle $C_n = \mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{+1,-1\})$

S - connected set of vertices

$$\varphi(S) = \frac{2}{2|S|} = \frac{1}{|S|}$$



$$\varphi(G) \approx \frac{2}{n}$$

$$\lambda_2 = 2\left(1 - \cos\frac{2\pi}{n}\right) = \Theta\left(\frac{1}{n^2}\right)$$

Tightness of Fiedler's Algorithm:

Produced a set S s.t

$$\underbrace{\varphi(S) \le \sqrt{2\lambda_2}}$$

But $\varphi(S) \ge \lambda_2/2$

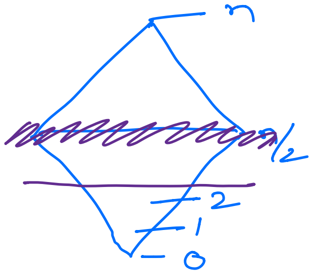
2^{nd} eigenvectors: characters of singleton sets

$$\left.\begin{aligned} \chi_{\{i\}} &: \{0,1\}^n \to \mathbb{R} \\ (x_1 \ldots x_n) &\mapsto (-1)^{x_i} \end{aligned}\right\} \ 2/n$$

$$\chi = \sum_{i \in n} \chi_{\{i\}} : \{0,1\}^n \to \mathbb{R} \ \Big\} \ 2^{nd} \text{ evector}$$

$(x_1 \dots x_n) \mapsto \sum_{i \in n} (-1)^{x_i}$

$= n - 2|x|$



$\uparrow$ increasing order of Hamming wt

Min cut
- majority cut
- $\Theta\left(\frac{1}{\sqrt{n}}\right)$

Expander Graphs:

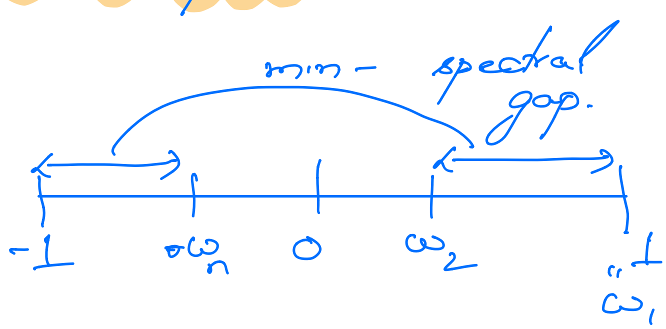
d-regular graphs. $\gamma \in [0, 1]$.

$G = (V, E)$ - d-regular is a γ-spectral expander. if $\omega(G) = \max\{\omega_2(G), |\omega_n(G)|\}$

spectral gap γ.

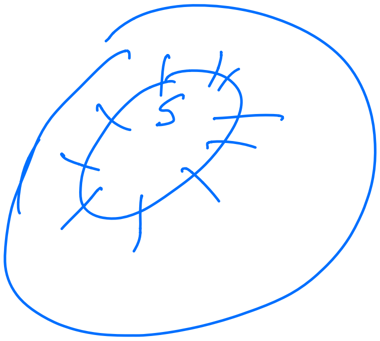
γ-expander

$\omega(G) \leq 1 - \gamma$



A family of d-regular graphs $\{G_n\}_{n=1}^{\infty}$ is a γ-spectral expander. ($\gamma = \Theta(1)$)

$\omega(G_n) \leq 1 - \gamma, \quad \forall n$



Cheeger Inequalities

- every cut in a γ-spectral expander has a constant fraction of edges going out

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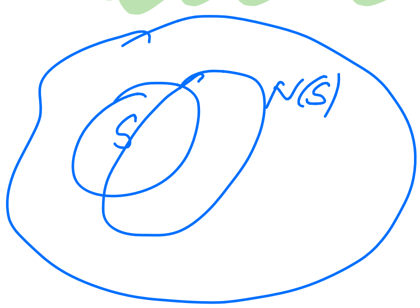
Vertex Expansion.

$G = (V, E)$ – d-regular graph is (K, A)-vertex expander (for $1 \le K \le |V|$, $A > 1$)

if $\forall\, S \subseteq V$

$$|S| \le K \Rightarrow |N(S)| \ge A|S|$$



$N(S) = \{ j \in V \mid \exists i \in S \; (i,j) \in E \}$

$(N^+(S) = S \cup N(S))$

Lemma: Spectral Expansion $\Rightarrow$ Vertex Expansion.

$G = (V, E)$ has spectral $\gamma = 1 - \omega$, then G is $\left(\rho N, \frac{1}{\omega^2(1-\rho)+\rho}\right)$-vertex expander for every $\alpha \in (0,1)$.

Pf:

Use Expander Mixing Lemma.

$$\left| \langle f, g \rangle_\pi - \mu_f \mu_g \right| \leq \omega \cdot \sigma_f \sigma_g \quad \ldots \quad (*)$$

$$\mu_f = \mathop{\mathbb{E}}_{i \sim \pi}[f(i)]$$

$$\sigma_f = \mathop{\mathrm{Var}}_{i \sim \pi}[f(i)]$$

$$= \mathop{\mathbb{E}}_{i \sim \pi}[f^2(i)] - \left(\mathop{\mathbb{E}}_{i \sim \pi}[f(i)]\right)^2$$

$S \subseteq V, \quad |S| < \rho n \qquad f = \mathbb{1}_S$

$T = V \setminus N(S) \qquad g = \mathbb{1}_T$

$\langle f, g \rangle_\pi = 0$

Let $|S| = \alpha n$; $|N(S)| = \beta n$; $|T| = (1-\beta) n$

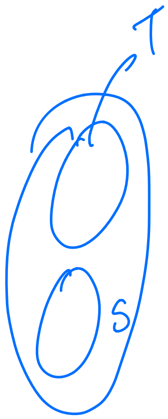
Apply EML.

$$\langle f, g \rangle_\pi = 0$$

$$\mu_f = \mathop{\mathbb{E}}_{i \sim \pi}[\mathbb{1}_S] = \pi(S) = \alpha$$

$$\mu_g = \pi(T) = 1 - \beta$$

$$\sigma_f = \alpha(1-\alpha) \quad ; \quad \sigma_g = \beta(1-\beta)$$



$$\alpha(1-\beta) \le \omega\sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

$$\alpha(1-\beta) \le \omega^2(1-\alpha)\beta$$

$$\beta\left[\omega^2(1-\alpha)+\alpha\right] \ge \alpha$$

$$\beta \ge \frac{\alpha}{\omega^2(1-\alpha)+\alpha} \ge \frac{\alpha}{\omega^2(1-\rho)+\rho}$$

$$= \frac{\alpha}{\alpha(1-\omega^2)+\omega^2} \quad \ge \qquad \text{since } \alpha \le \rho$$

G is $\left(\rho N, \frac{1}{\omega^2(1-\rho)+\rho}\right)$-expander for all $\rho \in [0,1)$.