Today

Lovasz Theta Function

CSS.205.1
Toolkit in TCS
- Lecture #28
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Recap from last lecture.

Shannon Capacity of a Graph

$G = (V, E)$ - simple graph (no self-loops & no multi-edges)

$$\Sigma(G) = \lim_{k \to \infty} \sqrt[k]{\alpha(G^{\boxtimes k})} = \sup_{k \to \infty} \sqrt[k]{\alpha(G^{\boxtimes k})}$$

where strong product $G_1 = (V_1, E_1)$; $G_2 = (V_2, E_2)$

$G_1 \boxtimes G_2 = (V, E)$

$V = V_1 \times V_2$

$((u_1, u_2), (v_1, v_2)) \in E \iff$ (i) $(u_1, u_2) \neq (v_1, v_2)$

(ii) $\forall i \in [2]$ $u_i = v_i$ or $(u_i, v_i) \in E$

Qn: What is $\Sigma(C_5)$?

Observations:

- $\alpha(G)$ — super multiplicative w.r.t strong prod

  (i.e. $\alpha(G \boxtimes H) \geq \alpha(G) \cdot \alpha(H)$)

- Suppose $f : \{\text{graphs}\} \to \mathbb{R}_{\geq 0}$ s.t

  (i) $\alpha(G) \leq f(G)$

  (ii) f is sub-multiplicative

  (i.e. $f(G \boxtimes H) \leq f(G) \cdot f(H)$)

  then $\Sigma(G) \leq f(G)$.

- Instantiating f with $\bar{\chi}(G)$ (or $\chi(\bar{G})$)

  $\Sigma(G) \leq \bar{\chi}(G)$

  Hence, $\sqrt{5} \leq \Sigma(G) \leq 3$.

Today: Lovasz's improved UB on $\Sigma(G)$.

Orthonormal Mappings

$i \mapsto v_i$ (unit vector) | No restriction on ambient dim

$\bar{u} = (u_1 \ldots u_n)$ — collection of unit vectors.

s.t $\{i,j\} \notin E \Rightarrow u_i \perp u_j$ (orthogonal)

$i \neq j$

Properties of Orthonormal Mappings

(1) $\bar{u}, \bar{v}$ - orthonormal mappings for G, H

$\Downarrow$

$\bar{u} \otimes \bar{v}$ - orthonormal mapping for $G \boxtimes H$

$\bar{u} = (u_1 \dots u_n) \in \mathbb{R}^d$, $\bar{v} = (v_1 \dots v_m) \in \mathbb{R}^e$

$\bar{u} \otimes \bar{v} = (\omega_{ij} \mid i \in [n], j \in [m]) \in \mathbb{R}^{de}$

$$\omega_{ij} = u_i \otimes v_j \qquad (\omega_{ij})$$

$$\omega_{ij}^{(rs)} = u_i^{(r)} \cdot \omega_j^{(s)}$$

$$\langle u_i \otimes v_j, u_k \otimes v_e \rangle = \langle u_i, u_k \rangle \cdot \langle v_j, v_e \rangle$$

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c - unit vector. (handle).

$$\vartheta(G, \bar{u}, c) = \max_{i \in [n]} \frac{1}{\langle c, u_i \rangle^2}$$

(2) $\alpha(G) \leq \vartheta(G, \bar{u}, c)$, $\forall$ orthonormal mappings $\bar{u}$ & handle c

Pf: $S \subseteq V$, S is an independent set $|S| \leq \alpha(G)$.

$$1 = \langle c, c \rangle \geq \sum_{i \in S} \langle c, u_i \rangle^2 \geq \left(\min_{i \in S} \langle c, u_i \rangle^2\right) \cdot |S|$$

$$\geq \left(\min_i \langle c, u_i\rangle^2\right) \cdot |S|$$

(3) $\vartheta(G \boxtimes H, \bar{u} \otimes \bar{v}, c \otimes d) = \vartheta(G, \bar{u}, c) \cdot \vartheta(H, \bar{v}, d)$

$$\langle u_i \otimes v_j, c \otimes d\rangle = \langle u_i, c\rangle \cdot \langle v_j, d\rangle$$

Lovasz' Theta Function:

$$\vartheta(G) = \min_{\bar{u}, c} \vartheta(G, \bar{u}, c)$$

(4) $\alpha(G) \leq \vartheta(G)$

(5) $\vartheta(G \boxtimes H) \leq \vartheta(G) \cdot \vartheta(H)$

Corollary: $\Sigma(G) \leq \vartheta(G)$.

Return to C_5 (5-cycle).

$\sqrt{5} \leq \Sigma(C_5) \leq \vartheta(C_5)$: What is $\vartheta(C_5)$?

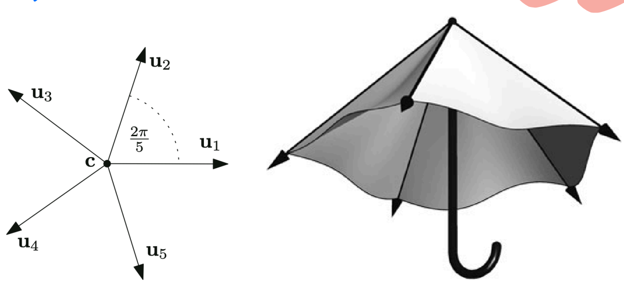

Fig. 3.3 A five-rib umbrella: fully open and viewed from the top (*left*), and partially folded so that non-adjacent ribs are perpendicular (*right*)

Candidate Orthonormal Mapping for C_5:

$k \in \{0, 1, \dots 4\}$

$$u_k = \frac{\left(\cos\frac{2\pi k}{5}, \sin\frac{2\pi k}{5}, z\right)}{\sqrt{\cos^2\frac{2\pi k}{5} + \sin^2\frac{2\pi k}{5} + z^2}} = \frac{\left(e^{\frac{i2\pi k}{5}}, z\right)}{\sqrt{1+z^2}}$$

$$c = (0, 0, 1).$$

Find z s.t. $\langle u_i, u_{i+2}\rangle = 0$

$$\langle u_0, u_2\rangle = 0$$

$$\langle u_0, u_2\rangle = \frac{\cos\frac{4\pi}{5} + z^2}{1+z^2} \; ; \quad \text{Hence}$$

$$z^2 = -\cos\frac{4\pi}{5} = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$$

$$\vartheta(C_5, \bar{u}, c) = \frac{1}{\langle c, u_0\rangle^2} = \frac{1+z^2}{z^2} = \frac{(\sqrt{5}+5)/4}{(\sqrt{5}+1)/4} = \sqrt{5}$$

Hence, $\vartheta(C_5) \leq \sqrt{5}$.

Thus, $\Sigma(C_5) = \sqrt{5}$.

$\boxtimes$

SDP Formulation for $\vartheta(G)$:

$$\vartheta(G) = \min_{\text{sym}} \lambda \quad \text{s.t.} \quad \begin{cases} M_{ij} = 1 \text{ if } i=j \text{ or } i \not\sim j \\ M \preceq \lambda I \end{cases}$$

(Pf uses the fact that $N \succeq 0$ (Pset 5) then $\exists\, v_1 \dots v_n$ st $N_{ij} = \langle v_i, v_j \rangle$)

Proof will yield an orthonormal mapping & handle st

$$\vartheta(G) = \frac{1}{\langle c, u_1 \rangle^2} = \frac{1}{\langle c, u_2 \rangle^2} = \qquad = \frac{1}{\langle c, u_n \rangle^2}.$$

Thm [Sandwich Theorem]

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G) = \chi(\bar{G})$$

Pf: Sufficient to prove.

$$\vartheta(\bar{G}) \leq \chi(G).$$

ie, SDP formulation for $\vartheta(\bar{G})$

$$\vartheta(\bar{G}) = \min_{\text{sym } M} \lambda$$

$$\begin{cases} M_{ij} = 1 & \text{if } i=j \text{ or } \{i,j\} \in E \\ M \preceq \lambda I \end{cases}$$

Suffices to show: Given a k-colouring of $\bar{G}$, there is a matrix M satisfying the above properties & $M \preceq kI$

k-simplex : $e_1 \dots e_k$ - unit vectors.

$$\tilde{u}_i = e_i - \frac{1}{k} \sum_{i=1}^{k} e_i$$

$$v_i = \frac{\tilde{v}_i}{\|\tilde{v}_i\|}$$

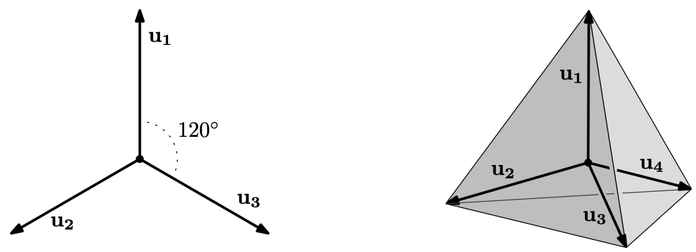

Fig. 3.4 Unit vectors with pairwise scalar products $-1/(k-1)$ for $k = 3, 4$

$\langle \tilde{v}_i, \tilde{v}_j \rangle$.

$$\tilde{v}_i = e_i - \frac{1}{k}\sum_{\ell=1}^{k} e_\ell = \left(1-\frac{1}{k}\right)e_i - \frac{1}{k}\sum_{\ell \neq i} e_\ell$$

$i = j$:
$$\|\tilde{v}_i\|^2 = \left(1-\frac{1}{k}\right)^2 + \frac{(k-1)}{k^2} = \frac{k^2-2k+1+k-1}{k^2} = \frac{k-1}{k}$$

$i \neq j$:
$$\langle \tilde{v}_i, \tilde{v}_j \rangle = \frac{-2}{k}\left(1-\frac{1}{k}\right) + \frac{(k-2)}{k^2} = \frac{-2k+2+k-2}{k^2} = -\frac{1}{k}$$

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{k-1} & \text{if } i \neq j \end{cases}$$

$\chi : V \to \{1, \dots, k\}$ is a k-coloring of $\bar{G}$

Vector Colouring.

Vertex i $\longrightarrow$ Vector $v_i \triangleq u_{\chi(i)}$

Vector Colouring satisfies.

(i) $\|v_i\| = 1$

(ii) $\{i,j\} \in E$, $\langle v_i, v_j \rangle = -\frac{1}{k-1}$

Define matrix $N \in \mathbb{R}^{V \times V}$

$$N(i,j) = \langle v_i, v_j \rangle$$

$$\begin{cases} N_{ii} = 1 \\ N_{ij} = -\frac{1}{k-1} & \text{if } \{i,j\} \in E(G) \end{cases}$$

Certainly $N \succeq 0$

Define: $M = kI - (k-1)N$

M satisfies

$$\begin{cases} \text{(i)} & M \text{ is symmetric} \\ \text{(ii)} & M_{ii} = k-(k-1) = 1 \\ \text{(iii)} & M_{ij} = -(k-1)\left(-\frac{1}{k-1}\right) \quad / \{i,j\} \in E \\ & \quad = 1 \\ \text{(iv)} & M \preceq kI. \quad (\text{i.e., } kI - M \succeq 0) \end{cases}$$

Hence, $\vartheta(\bar{G}) \leq k$.

$\overline{\vartheta(\bar{G})}$; $M \longrightarrow N = \frac{1}{\lambda-1}(\lambda I - M)$ $\boxtimes$

$\vartheta(\bar{G})$ - yields a vector colouring of G s.t

$$\left.\begin{aligned} &\forall i, \quad \|v_i\| = 1 \\ &\forall i \sim j, \quad \langle v_i, v_j \rangle = -\frac{1}{\lambda - 1} \end{aligned}\right\}$$

Further properties of $\vartheta(G)$.

① $\vartheta(G) = \max\limits_{\text{sym} B} \operatorname{Tr}(BJ)$ (J is all 1's matrix)

$$\begin{cases} B \succeq 0 \\ B_{ij} = 0 \text{ if } i \neq j \ \& \ i \not\sim j \\ \operatorname{Tr}(B) = 1 \end{cases}$$

② $\vartheta(G) = \max\limits_{\substack{\bar{u}, c \\ \text{orth map}}} \sum\limits_{i=1}^{n} \langle c, u_i \rangle^2$

③ $\vartheta(G) = \max\limits_{A} \left(1 - \frac{\lambda_1(A)}{\lambda_n(A)} \right)$

$$\begin{cases} A_{ij} = 0 \text{ if } i \sim j \\ \lambda_1(A) \geq \dots \geq \lambda_n(A) \end{cases}$$

– non-decreasing order

Hoffman Bound:

A - adjacency matrix. $\chi(G) \geq 1 - \frac{\mu_1}{\mu_n}$.

Pf. $\chi(G) \geq \vartheta(\bar{G}) \geq 1 - \frac{\mu_1}{\mu_n}$. $\boxtimes$

Sandwich Theorem: $\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$

Perfect Graphs: G is perfect if $\forall$ induced graphs G', $\alpha(G') = \bar{\chi}(G')$

Weak Perfect Graph Thm: G is perfect $\Leftrightarrow$ $\bar{G}$ is perfect