Today - Spectral Expansion

- Expander Mixing Lemma

- Vertex vs Spectral Exp

CSS.413.1 Pseudorandomness
Lecture 08 (2021-9-16)
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Today: Expansion in terms of spectrum of the (normalized) adjacency matrix.

$G = (V, E)$ on N vertices.
(typically D-regular)

A - adjacency matrix $\quad A \in \mathbb{R}^{V \times V}$

$$A(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{otherwise.} \end{cases}$$

A is symmetric.

M - (Normalized) Adjacency Matrix [Random Walk Matrix]

$$M(u,v) = A(u,v) / \deg(u)$$

Row Stochastic matrix
non-negative.

Equivalently $M = D^{-1}A$; $D = \mathrm{Diag}(\deg)$

Linear Operator: $\mathbb{R}^V \longrightarrow \mathbb{R}^V$

(Left Multiplication) $f \longmapsto Mf$

$f \in \mathbb{R}^V$ (vectors - column vectors)

$$(Mf)(u) = \sum_v M(u,v) f(v) \quad , \forall u \in V.$$

"Averaging" Operator.

$$M \cdot \mathbb{1} = \mathbb{1}$$

Right Multiplication

$p \in \mathbb{R}^V$

$p^T \longmapsto p^T M$

$p \longmapsto M^T p$

$$(M^T p)(u) = \sum_v M^T(u,v) p(v)$$

$$= \sum_v M(v,u) p(v)$$

$$(M^T p)(u) = \sum_v p(v) M(v,u)$$

p - prob dist on vertices.

Random walk spectral by Matrix

- Represents Random walk

$p^T \to p^T M \to p^T M^2 \to \cdots \to p^T M^n \to \cdots$

Is there a stationary dist?

i.e., $\pi^T = \pi^T M$

G is D-regular, then the uniform distribution

i.e., $\pi(u) = \frac{1}{N}$, $\forall u \in V$

is a stationary distribution.

More generally $\pi(u) = \frac{\deg(u)}{\sum_{w \in V} \deg(w)}$

is a stationary dist for M

i.e., $\pi^T M = \pi$

Eigenvalues & Eigenvectors:

M is a $\mathbb{R}^{v \times v}$-matrix.

$\varphi \in \mathbb{R}^v \setminus \{0\}$, $\lambda \in \mathbb{R}$ such that

$$M\varphi = \lambda \varphi$$

φ is an eigenvector w/ eigenvalue λ

Quadratic Form

Inner Product: $\langle , \rangle : \mathbb{R}^v \times \mathbb{R}^v \to \mathbb{R}$.

bilinear 2. $\langle f, f \rangle \geq 0$ & $= 0$ iff $f = 0$.

$f, g \in \mathbb{R}^V$

$$\langle f, g \rangle_\pi = \sum_{u \in V} \pi(u) f(u) g(u)$$

$$= \mathbb{E}_{u \leftarrow_\pi V} [f(u) g(u)]$$

Quadratic form associated w/ M

$$\langle f, Mg \rangle_\pi = \mathbb{E}_{u \leftarrow_\pi V} [f(u) (Mg)(u)]$$

$$= \mathbb{E}_{u \leftarrow_\pi V} \left[f(u) \sum_{v \in V} M(u,v) g(v)\right]$$

$$= \mathbb{E}_{u \leftarrow_\pi V} \left[\sum_{v \in V} M(u,v) f(u) g(v)\right]$$

$$= \mathbb{E}_{e=(u,v) \leftarrow_\omega E} [f(u) g(v)]$$

$$= \langle g, Mf \rangle_\pi = \langle Mf, g \rangle_\pi$$

M is self-adjoint w.r.t. $\langle \, , \rangle_\pi$

(i.e., $\langle f, Mg \rangle_\pi = \langle Mf, g \rangle_\pi$)

<u>Spectral Theorem:</u> Let $M \in \mathbb{R}^{N \times N}$ & $\langle , \rangle$ be an inner product s.t M is self-adjoint w.r.t $\langle , \rangle$ then there exist

$$v_1 \; \ldots \; v_n \in \mathbb{R}^N$$

$$\lambda_1 \; \ldots \; \lambda_n \in \mathbb{R}$$

s.t (i) $\forall i$, $Mv_i = \lambda_i v_i$

(ii) $\langle v_i, v_j \rangle = \mathbb{1}[i=j]$

Hence, M (ie, random walk matrix) has a full eigen decomposition.

Remark: ① $M\mathbb{1} = \mathbb{1}$ (ie, $\mathbb{1}$ is an e.vector w/ e.value 1)

② M: -averaging operator.

$Mv = \lambda v$ for some $\lambda \in \mathbb{R}$ & $v \in \mathbb{R}^V$

then $|\lambda| \leq 1$

M- random walk

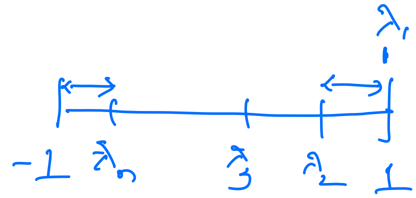
$1 = \lambda_1, \geq \lambda_2 \quad \ldots \quad \geq \lambda_n \geq -1$ - evalues

$\mathbb{1}, \quad v_2 \ldots \quad v_n.$ - evectors

v_i are pairwise orthogonal

$\langle v_i, \mathbb{1} \rangle_\pi = 0.$

Eigen Spectrum:



Spectral gap $= \min\{1-\lambda_2, \ 1-|\lambda_n|\}.$

Spectral Expansion: A graph $G=(V,E)$ on N-vertices is said to be a γ-spectral expander if spectral gap of the associated random walk matrix is at least γ.

$\|f\|_{2,\pi}^2 = \langle f, f \rangle_\pi$

Equivalently, spectral gap

$$r(M) = 1 - \max_{f \perp_\pi \mathbb{1}} \frac{\|Mf\|}{\|f\|}$$

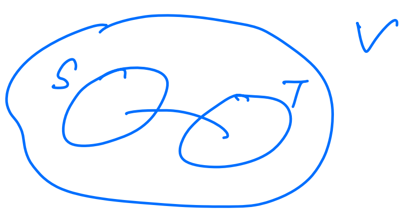
(consequence to spectral theorem)

Spectral Expansion vs Vertex Expansion

Expander Mixing Lemma: (EML)

M - random walk matrix with spectral gap $1-\lambda$, and $S, T \subseteq V$, $\pi(S) = \alpha$, $\pi(T) = \beta$.

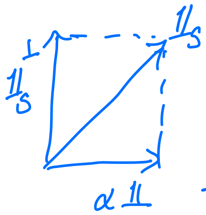
$$\left| \Pr_{e=(u,v)\sim E}[u \in S, v \in T] - \alpha\beta \right| \le \lambda \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$



$$\le \lambda\sqrt{\alpha\beta}$$

Pf: $\mathbb{1}_S$ - indicator vector for set S

$$\mathbb{1}_S(u) = \mathbb{1}[u \in S]$$



$\mathbb{1}_S = a\mathbb{1} + \mathbb{1}_S^\perp$ where $\mathbb{1}_S^\perp \perp \mathbb{1}$

$\langle \mathbb{1}_S, \mathbb{1} \rangle_\pi = a\langle \mathbb{1}, \mathbb{1} \rangle \Rightarrow \alpha = a \cdot 1$

Hence, $\mathbb{1}_S = \alpha \mathbb{1} + \mathbb{1}_S^{\perp}$

Similarly $\mathbb{1}_T = \beta \mathbb{1} + \mathbb{1}_T^{\perp}$

$$\Pr_{e\sim(u,v)\sim E}[u \in S, v \in T] = \langle \mathbb{1}_S, M\mathbb{1}_T \rangle_\pi$$

$$= \langle \alpha\mathbb{1} + \mathbb{1}_S^{\perp}, M(\beta\mathbb{1} + \mathbb{1}_T^{\perp}) \rangle_\pi$$

$$= \alpha\beta \langle \mathbb{1}, \mathbb{1} \rangle_\pi + \alpha \langle \mathbb{1}, M\mathbb{1}_T^{\perp} \rangle_\pi$$
$$+ \beta \langle \mathbb{1}_S^{\perp}, M\mathbb{1} \rangle_\pi + \langle \mathbb{1}_S^{\perp}, M\mathbb{1}_T^{\perp} \rangle_\pi$$

$$= \alpha\beta + \alpha \langle M\mathbb{1}, \mathbb{1}_T^{\perp} \rangle +$$
$$\beta \langle \mathbb{1}_S^{\perp}, \mathbb{1} \rangle_\pi + \langle \mathbb{1}_S^{\perp}, M\mathbb{1}_T^{\perp} \rangle_\pi$$

$$= \alpha\beta + \langle \mathbb{1}_S^{\perp}, M\mathbb{1}_T^{\perp} \rangle_\pi$$

$$\left| \Pr_{(u,v)\sim E}[u \in S, v \in T] - \alpha\beta \right| = \left| \langle \mathbb{1}_S^{\perp}, M\mathbb{1}_T^{\perp} \rangle_\pi \right|$$

$$\leq \|\mathbb{1}_S^{\perp}\|_{2,\pi} \cdot \|M\mathbb{1}_T^{\perp}\|_{2,\pi}$$

$$\leq \|\mathbb{1}_S^{\perp}\|_2 \cdot \lambda \|\mathbb{1}_T^{\perp}\|_{2,\pi}$$

What is $\|\mathbb{1}_S^{\perp}\|_2$

$$\mathbb{1}_S = \alpha \mathbb{1} + \mathbb{1}_S^{\perp}$$

$$\|\mathbb{1}_S\|_2^2 = \|\alpha \mathbb{1}\|_2^2 + \|\mathbb{1}_S^\perp\|_2^2$$

$$\langle \mathbb{1}_S, \mathbb{1}_S \rangle_\pi = \alpha^2 \langle \mathbb{1}, \mathbb{1} \rangle_\pi + \|\mathbb{1}_S^\perp\|_2^2$$

$$\alpha = \alpha^2 + \|\mathbb{1}_S^\perp\|_2^2$$

$$\|\mathbb{1}_S^\perp\|_2 = \sqrt{\alpha(1-\alpha)}$$

$$|\mathrm{Diff}| \le \lambda \, \|\mathbb{1}_S^\perp\|_{2,\pi} \cdot \|\mathbb{1}_T^\perp\|_{2,\pi}$$

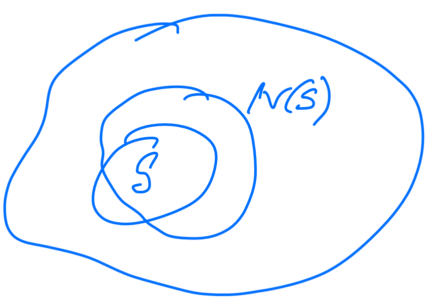
$$= \lambda \sqrt{\alpha(1-\alpha)\,\beta(1-\beta)}$$

$\boxtimes$

(Partial Converse: [Bilu Linial]
EML conclusion $\Rightarrow$ spectral gap)

$|\Pr - \alpha\beta| \le \theta\sqrt{\alpha\beta} \Rightarrow \lambda \le C\,\theta\left(1 + \log\frac{1}{\theta}\right)$

Spectral Expansion $\Rightarrow$ Vertex Expansion



S, $\pi(S) = \alpha$

$N(S)$. $\pi(N(S)) = \beta$

$T = V \setminus N(S)$

Apply EML to sets S & T.

$$\left| \Pr_{(u,v)\sim E}[u \in S, v \in T] - \alpha(1-\beta) \right| \le \lambda \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

$$\Rightarrow \alpha(1-\beta) \le \lambda \sqrt{\alpha(1-\alpha)\beta(1-\beta)}$$

$$\Rightarrow \alpha(1-\beta) \le \lambda^2 \beta(1-\alpha)$$

$$\Rightarrow \beta \ge \frac{\alpha}{\lambda^2(1-\alpha)+\alpha}$$

ie, $\pi(S)=\alpha$, then $\pi(N(S)) \ge \dfrac{\alpha}{\lambda^2(1-\alpha)+\alpha}$

$$= \alpha\left(\frac{1}{\alpha(1-\lambda^2)+\lambda^2}\right)$$

Sets of size αn expand by $\dfrac{1}{\alpha(1-\lambda^2)+\lambda^2}$ - factor

$- \alpha / \alpha(1-\lambda^2)+\lambda^2$

<u>Thm</u>: $G=(V,E)$ is a N-vertex D-regular graph w/ spectral gap. $\gamma = 1-\lambda$, then for all $\rho \in (0,1)$. for all sets S of size at most ρN, $N(S) \ge \dfrac{|S|}{\lambda^2(1-\rho)+\rho}$.

ie, G is $\left(\rho N, \dfrac{1}{\lambda^2(1-\rho)+\rho}\right)$-vertex expander.

Alternate viewpoint of EML.

$f, g : V \to \mathbb{R}$

$$\left| \mathop{\mathbb{E}}_{(u,v)\sim E}[f(u)g(v)] - \mu_f \mu_g \right| \le \lambda\, \sigma_f \cdot \sigma_g$$

(where means, s.d is computed w.r.t π)

$$\mu_f = \mathbb{E}_{\omega \sim \pi}[f(\omega)]$$

$$\sigma_f^2 = \mathbb{E}_{\omega \sim \pi}[f^2(\omega)] - \left(\mathbb{E}_{\omega \sim \pi}[f(\omega)]\right)^2$$