More intuition for zig-zag products.

$G = (N, D, \lambda)$ expander.

$v_1, v_2, \ldots, v_t \in G.$

$a_1, a_2, a_3, \ldots, a_{t-1} \rightarrow t-1$ indep. samples from $[D]$.

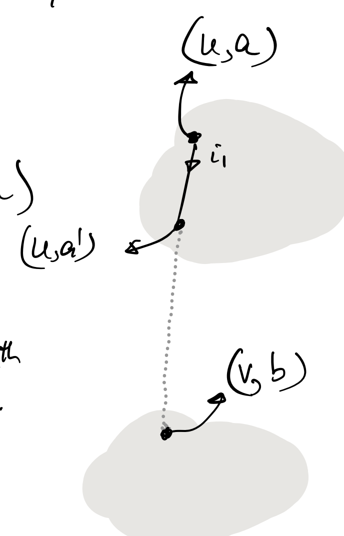
Idea: Take H to be a (D, d, λ')-expander.

$$\begin{array}{cccc} b_1 & b_2 & b_3 & \ldots & b_{t-1} \\ & \| & \| & & \\ & \Gamma_H(b_1, i_1) & \Gamma_H(b_2, i_2) & & \end{array}$$

Qn: Can we build a graph $G * H$ that "mimics" this walk?

Want v_3's i_2-th neighbour to be v_4.

Fix: Let vertices keep track of the b's.



$G * H =$ on $[N] \times [D]$ $\quad (u, a)$

Say the i^{th} neighbour$_{G*H}$ of (u, a)

$= Rot_G(u, a')$ where a' is the i^{th} neigh.

Hope: this works.

$I \otimes H \cdot Rot_G.$

Fix: Add another intra-cloud step.

$I \otimes H \cdot Rot_G \cdot I \otimes H$ — Zigzag prod.

Pseudorandomness : Lecture 12.

Instructor: Ramprasad
Date: 2021-09-30

Agenda:
- Finishing up some expander family constructions
- "low space" algorithms for undirected s-t connectivity.

Recap: - Graph operations:

	Good	Bad.
- Powering -	reduces λ.	inc. degree
- Tensoring -	increases #vertices	inc. degree
- Zigzag -	reduces degree	slightly worsens λ.

Base graph: $(D^4, D, 1/8)$ - expander H

$$G_1 = H^2. \qquad G_t = G_{t-1}^2 ⓩ H.$$

Claim: G_t is a $(D^{4t}, D^2, 1/2)$-expander.

Pf:
$$G_{t-1}^2 ⓩ H = (D^{4(t-1)}, D^4, 1/4) ⓩ (D^4, D, 1/8)$$
$$= (D^{4t}, D^2, \lambda)$$

$$(1-\lambda) \geq \frac{3}{4} \cdot \frac{7}{8} \cdot \frac{7}{8} \approx 0.52 \qquad \square.$$

$$\text{Time}(t) \leq 2 \cdot \text{Time}(t-1) + O(1) = \exp(t)$$
$$= \text{poly}(|G_t|)$$

"time" to compute Rot_{G_t}

Fully explicit construction:

Base graph $H = (D^8, D, 1/8)$-expander.

$$G_1 = H^2$$
$$G_t = \left(G_{\lceil t/2 \rceil} \otimes G_{\lfloor t/2 \rfloor}\right)^2 ⓩ H.$$

Claim: G_t is a $(D^{8t}, D^2, 1/2)$ expander for all $t \geq 1$
Also, Rot_{G_t} computable in poly(t) time.

G - undirected graph. s, t vertices.
Is there a path from $s \to t$? DFS, BFS?
What if you only have $O(\log n)$ space. # vertices in G.

walks $\leq D^{O(\log n)} = \mathrm{poly}(n)$.

Randomised Algo:
Start from s and take a random walk
for $\ell = n^6$ steps. can be made $O(\log n)$ if G was an expander. (const. deg)
If we encounter t, return "Yes".
Else, return "No".

Why would this algo work? It actually does!

Ex: (in PS2) For any connected, non-bipartite d-regular n-vertex graph, $\lambda(G) \leq 1 - \frac{1}{2d^2n^2}$.

$$x = [0 \; - \cdots \; 0 \; \underset{s}{1} \; 0 \; - \cdots \; 0] \xrightarrow{M^\ell} [\quad \underset{t}{\mid\;\mid} \quad] \; y$$

$$\|y - \pi\|_2 = \|x M^\ell - \pi\|_2 = \|(x - \pi) \cdot M^\ell\| \leq \lambda^\ell \cdot \|x - \pi\|_2 \leq \frac{1}{n^2}$$

$\Rightarrow y$ puts $\geq \frac{1}{n} - \frac{1}{n^2}$ mass on t.

What should ℓ be so that $\lambda^\ell \leq 1/n^2$

$$\left(1 - \frac{1}{2d^2n^2}\right)^\ell \leq 1/n^2 \qquad \ell = \Theta(d^2 n^2 \log n)$$

" you should expect to see t about n^2 times in an n^6 length RW".

Qn: What if G had spectral gap $\gamma > 0$?

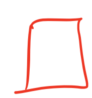
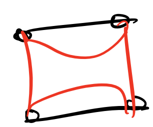
What should l be?

Want $\lambda^l \leq 1/n^2 \Rightarrow l = O(\log n)$ if $\lambda < 1$ is a constant.

Qn: Can we solve s.t connectivity for general undirected graphs in $O(\log n)$ space?

" If only we could make G an expander"

Attempt: Slap an expander onto G.

Lose disconnectivity completely!  

What would happen if we just tried "expanderising" G using the square, tensor & zig-zags?

Each component becomes an expander.

$\Rightarrow$ disconnectivity is not lost.

With all details:

Step 1: Make G D^2-regular (for const D) without altering components

D^2-reg graph.

$s \rightsquigarrow t \iff (s,1) \rightsquigarrow (t,1)$

$s \rightsquigarrow t$ in G iff $s_1 \rightsquigarrow t_1$ in G_1 — (n', D^2)

For $i = 2, \ldots, l = O(\log n)$:

$G_i = G_{i-1}^2 \textcircled{z} H$ $(D^4, D, 1/4)$

s_k and t_k are arbitrary vertices within cloud(s_{k-1}) & cloud(t_{k-1})

G_l is a graph on $n \cdot D^{4l}$ vertices $= \text{poly}(n)$.

deg $= D^2$.

Lemma 1: G_l, for $l = O(\log n)$, has all components having const. spectral gap.

Lemma 2: Rot_{G_l} can be computed in $O(\log n)$ space (given only the adj. matrix of G)

Pf of lemma 1:

spectral gap increases by a constant factor each time... unless it is already high.

Claim: $\gamma(G_k) \geq \min\left\{ \frac{35}{32} \cdot \gamma(G_{k-1}), \frac{1}{18} \right\}$.

Pf: $\gamma(G_k) \geq \left(1 - (1 - \gamma(G_{k-1}))^2\right) \cdot \frac{3}{4} \cdot \frac{3}{4}$

$$= \frac{9}{16}\left(2\gamma_{k-1} - \gamma_{k-1}^2\right)$$

$$= \underbrace{\frac{9}{16} \cdot (2 - \gamma_{k-1}) \cdot \gamma_{k-1}}$$

$\leq 1/18 \Rightarrow \frac{9}{16} \cdot (2-\gamma_{k-1}) \cdot \gamma_{k-1} \leq \frac{1}{18} \quad \frac{16}{18 \times 9}$

$$\frac{9}{16} \cdot (2-\gamma_{k-1}) \geq \frac{35}{32}$$

$\square$

$$\gamma_0 = \frac{1}{\mathrm{poly}(n)} \longrightarrow \gamma_l = \min\left(\left(\frac{35}{32}\right)^l \cdot \frac{1}{\mathrm{poly}(n)}, \frac{1}{18}\right)$$

$\Rightarrow$ in $l = O(\log n)$ steps, $\gamma_l \geq \frac{1}{18}$

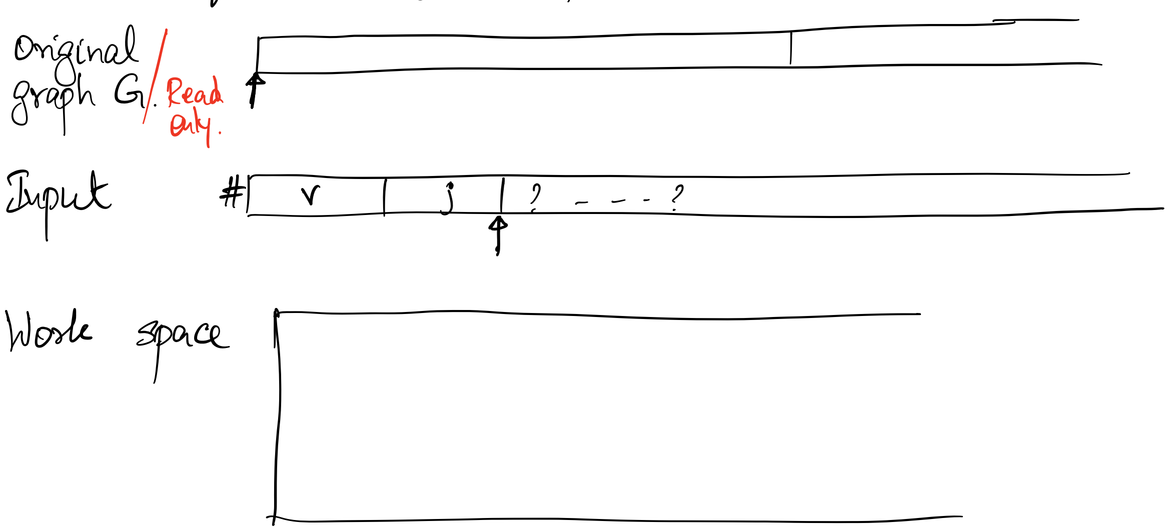
Computing Rot_{G_l} in $O(\log n)$ space.

Claim: If we can compute Rot_{G_k} in space s

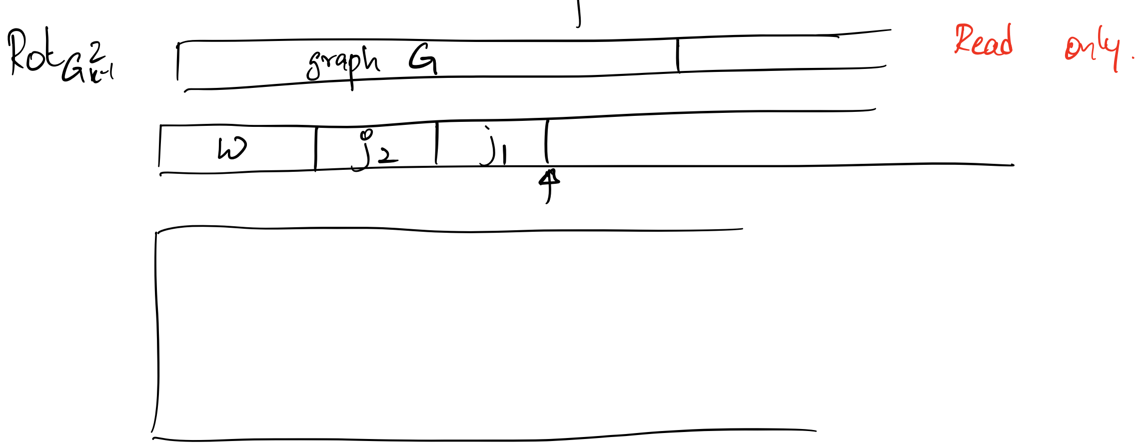
then we can compute $Rot_{G_{k+1}}$ in space $S + O(1)$.

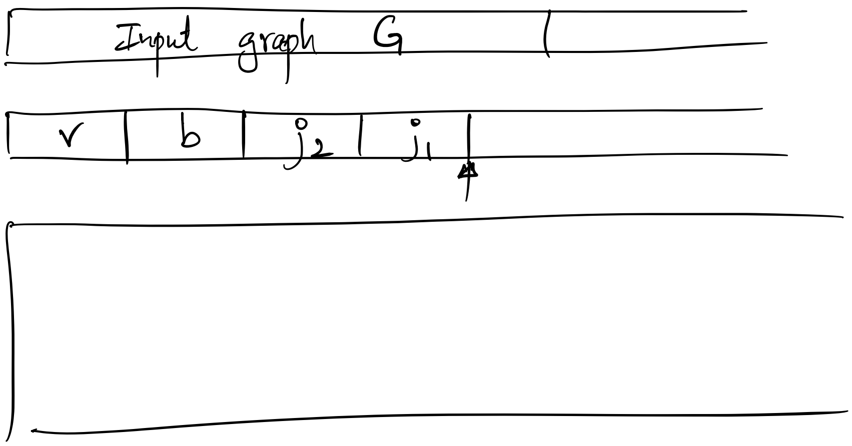
Pf: $G_{k+1} = G_k^2 \; z \; H$.

Specification of computation:



Output expected on tape 2, with same head position, rest intact & workspace cleared.





Read only.

$\Gamma_H(a, i_1) = (a', j_1)$

$\Gamma_{G_{k-1}^2}(u, a') = (v, b')$

$\Gamma_H(b', i_2) = (b, j_2)$

Space for $\mathrm{Rot}_{G_k} \le$ Space for $\mathrm{Rot}_{G_{k-1}} + O(1)$ $\square$

Thm [Reingold] U-ST-CONN $\in$ LOGSPACE.

Path enumeration algo: