Pseudorandomness – Lecture 13.

Agenda: – Spectral sparsifiers.
[Spielman & Srivastava].

Given a graph G on n-vertices & m edges.
We may want to get a sense of some connectivity structure: – "max flows" = min cut
"clusters"
"electrical flows".
"counting triangles".

Issue: # edges could be $\Theta(n^2)$.

Sparsification: G (n vertices, m edges) $\longrightarrow$ H (n vertices, $\tilde{O}(n)$ edges (weighted))
"preserves most info".

Cut sparsifiers [Benczur-Karger]
H ε-cut sparsifies G if for any $S \subseteq [n]$
$E_G(S, \bar{S}) \approx_{1\pm\varepsilon} E_H(S, \bar{S})$.
(sum of the weights of edges).

[Benczur-Karger] Any G has an ε-cut sparsifier of size $O(n \log n / \varepsilon^2)$. Can also be found very quickly.

A stronger notion of approximation:

$x \in \mathbb{R}^n$. $\mathcal{E}_G(x) = \sum_{(i,j) \in G} (x_i - x_j)^2$ → Variance across edges.

→ spring/resistance network.

"energy of the network G with potentials x"

Defn: H is a K-spectral approx of G if $\forall x \in \mathbb{R}^n$

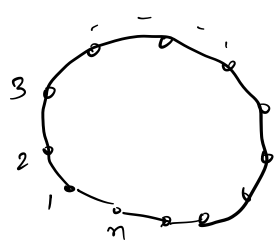
$$\mathcal{E}_H(x) \le \mathcal{E}_G(x) \le K \cdot \mathcal{E}_H(x).$$

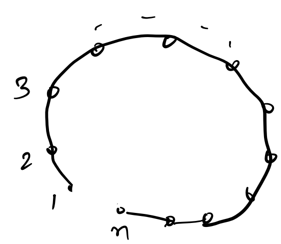
Obs: Any spectral approximator is a cut approximator.

Pf: For a set S, if $x = \mathbb{1}_S$

then $\mathcal{E}_G(x) = \text{cut}(S, \bar{S})$. $\square$.

But is a stronger notion:





$$\mathcal{E}_G(x) = 1 \cdot (n-1) + n^2. \quad \gg \quad \mathcal{E}_H(G) = 1 \cdot (n-1)$$

An important concept: the Laplacian.

$L_G = D_G - A_G$ → adjacency matrix.

D_G → diag matrix of degrees

Ex: (Pset 2) $x^T L_G x = \mathcal{E}_G(x)$

Cor: $x^T L_G x \geq 0$ for any $x \in \mathbb{R}^n$

$\therefore$ L_G is positive semi-definite (PSD).

$L_G \succcurlyeq 0$

Defn: For symmetric matrices A, B.

$$A \succcurlyeq B \iff A - B \succcurlyeq 0.$$

Fact: If H is a κ-spectral approximator of G, then all eigenvalues of L_G are within κ-factor of those of L_H.

... and hence the name spectral sparsifiers.

Thm: [Spielman-Srivastava]. For any graph G, there is a $(1+\varepsilon)$-spectral sparsifier with $\leq O(n \log n \cdot \frac{1}{\varepsilon^2})$ edges. (Can also be found in near linear time).

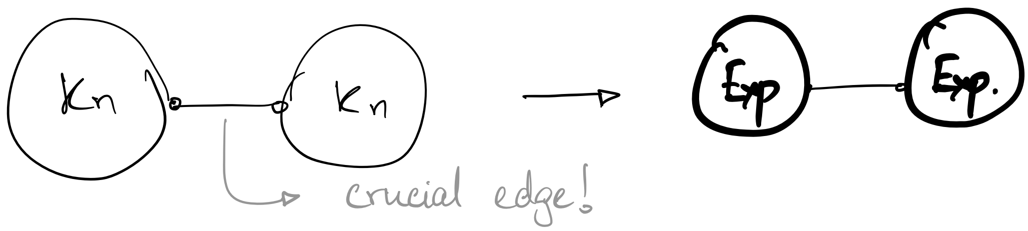
How do we find a spectral sparsifier?

Attempt: Can we randomly sample edges?

Test case 1: Complete graph $\longrightarrow$ expander.

Works.

Test case 2: Dumbbell



Not all edges are equally important!
Uniformly sampling edges is bad!

Need some way of figuring out how crucial an edge is.

$$x^T L_G x = \sum_{(u,v)\in G} (x_u - x_v)^2 = \sum_{(u,v)\in G} x^T L_{uv} x$$

$$L_{uv} = \begin{array}{c|cc} & u & v \\ \hline u & 1 & -1 \\ v & -1 & 1 \end{array}$$

$$= \sum x^T (e_u - e_v)(e_u - e_v)^T x$$

$$\therefore \quad L_G = \sum_{e\in G} b_e b_e^T \qquad b_e = e_u - e_v$$

Want to find H: $\quad L_H = \sum s_e \cdot b_e b_e^T$ ⟶ wt of e in H.

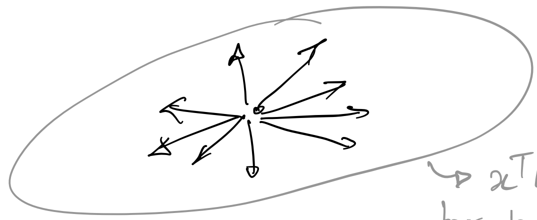
$$\text{s.t.} \quad L_G \preceq L_H \preceq K \cdot L_G.$$

Matrix Chernoff Bound [Rudelson, Alswede-Winter, Tropp]
Suppose X is a $d\times d$ matrix valued RV with $0 \preceq X \preceq M\cdot I$ and $\mathbb{E}[X] = I$.
If $X_1, \ldots, X_k$ are independent samples of X, then

$$\Pr\left[(1-\varepsilon) I \preceq \frac{1}{k}\cdot\sum_{i=1}^{k} X_i \preceq (1+\varepsilon) I\right] \leq 2d \cdot \exp\left(-\frac{\varepsilon^2 \cdot k}{4M}\right).$$

(very similar to the usual chernoff bound).

Geometric view: quadratic forms as ellipsoids.



$$M L_G M \preccurlyeq M L_H M \preccurlyeq K \, M L_G M \qquad M \text{ any PSD matrix.}$$

For $M = L_G^{-1/2}$

$$L_G = \sum \lambda_i u_i u_i^T$$

$$L_G^{-1/2} = \sum \frac{1}{\sqrt{\lambda_i}} \cdot u_i u_i^T$$

$$I \preccurlyeq \tilde{L}_H \preccurlyeq K \cdot I$$

$$L_G = \sum b_e b_e^T \Rightarrow M L_G M = \sum M \cdot b_e b_e^T M$$

$$= \sum_{e \in G} v_e v_e^T$$

where $v_e = L_G^{-1/2} \cdot b_e$.

Rephrasing our question:

Given m vectors $\{v_e : e \in G\}$ from $\mathbb{R}^n$ s.t $\sum v_e v_e^T = I$.

find a sparse subset of these and weights s.t

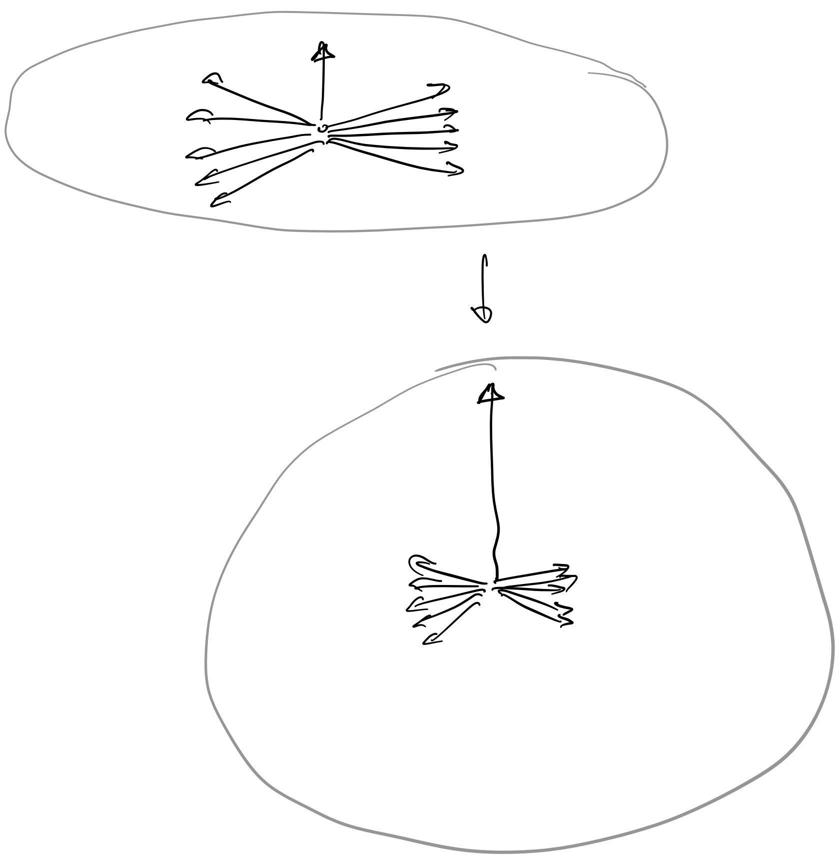
$$I \preccurlyeq \sum s_e \cdot v_e v_e^T \preccurlyeq (1+\varepsilon) \cdot I$$

Revisiting our examples:

1) Complete graph K_n:

Doesn't change the picture at all.

Dumb-bell graph:



Really stretches out crucial edges!

$\|v_e\|^2$ - proxy for how important the edge e is.

What is $\|v_e\|$? $\qquad \| L_G^{-1/2} b_e \| = b_e^T L_G^{-1} b_e$

"effective resistance of edge e".

Now we can do random sampling!

$\sum v_e v_e^T = I$. Let X be a random matrix that is $\frac{v_e v_e^T}{p_e}$ with prob p_e.

(to be defined)

so that $\mathbb{E}[X_i] = I$.

Matrix Chernoff Bound [Rudelson, Alswede-Winter, Tropp]

Let $X_1, \dots, X_k$ be i.i.d random $d \times d$ matrices with $0 \preceq X_i \preceq M \cdot I$, $\mathbb{E}[X_i] = I$.

Then $\Pr\left[\left\| \frac{1}{k} \sum X_i - I \right\| \geq \varepsilon \right] \leq 2d \cdot \exp\left(- \frac{k \cdot \varepsilon^2}{4M} \right)$

$\leq 1/3$

if $k = O\left(\frac{M}{\varepsilon^2} \log d \right)$

$X_e = \frac{v_e v_e^T}{p_e}$ with prob p_e

want to minimise $M = \max_e \frac{\| v_e v_e^T \|}{p_e} = \max \frac{\|v_e\|^2}{p_e}$.

Optimal: set $p_e \propto \|v_e\|_2^2$

$\sum p_e = 1$ $\quad\quad$ $\sum \|v_e\|^2 = \sum Tr(v_e v_e^T)$

$= Tr\left(\sum v_e v_e^T\right) = n.$

$\therefore\ p_e = \dfrac{\|v_e\|^2}{n}$, $\quad M = n$ in this case.

Matrix chernoff says. $O\left(\frac{n}{\varepsilon^2} \log n\right)$ samples enough

All of this can actually done in near-linear time!

Final Algo:

- ▷ Compute $M = L_G^{-1/2}$
- ▷ For each edge $e = (u,v)$, compute $p_e = \dfrac{\|M(e_u - e_v)\|_2^2}{n}.$
- ▷ Set $H = \phi$
- ▷ For $i = 1, \dots, k = O(n \log n \cdot \frac{1}{\varepsilon^2})$:
  - Sample an edge from G acc. measure $\{p_e\}$.
  - Add the edge to H.
- ▷ Return H.

actually $O(m\sqrt{\log m}\ \text{polyloglog}(m))$ time!

Can actually be done in near-linear time "$Lx = b$" by Nisheeth Vishnoi.

[Batson-Spielman-Srivastava] - With just $O(n/\varepsilon^2)$ edges.
(interlacing)

[Marcus-Spielman-Srivastava] "Interlacing polynomials"

Eventually solved the $\underbrace{\text{Kadison-Singer}}$ problem & a host of other problems!

a statement in quantum mechanics.

Weaver's Conjecture (equiv to KS).

Given vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ s.t $\sum v_i v_i^T = I$ and $\|v_i\|_2^2 \leq \delta$ $\forall i$. Does there exist a partition $[m] = S_1 \cup S_2$ s.t.

$$\Big\| \sum_{i \in S_b} v_i v_i^T \Big\| \preccurlyeq (1-\eta) \cdot I \qquad ?$$

[MSS]: Yes! In fact. $(\frac{1}{2}-\varepsilon) \cdot I \preccurlyeq \| \sum_{i \in S_b} v_i v_i^T \| \preccurlyeq (\frac{1}{2}+\varepsilon) \cdot I$ for a constant $\varepsilon = O(\sqrt{\delta})$.