Today

- High Dimensional Expanders

CSS.413.1
Pseudorandomness
Lecture 27 (2021-12-7)
Instructor: Prahladh Harsha.

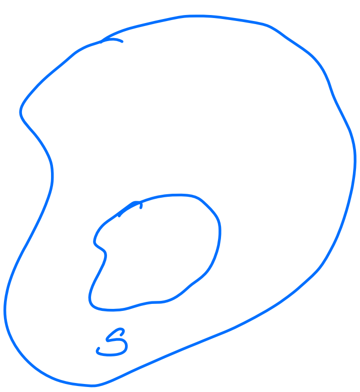
- High-dimensional expansion
  - gen of expansion in graphs to hypergraphs

Recap

Expander Graphs.

- Definition

$G=(V,A)$



(i) Spectral Expansion

A - adj matrix of a graph. (normalized).

$\lambda(A) = \max\{\lambda_2, |\lambda_n|\}$

$\lambda \ll 1$

(ii) Combinatorial Expansion

$\forall S,\ |S| \le K,\quad |N(S)| \ge \alpha |S|$

(then G is (k,α)-expander.

(iii) Random Walk

- random walk converges to the uniform (stationary) dist fast

- A - normalized adj matrix

J/n - $\begin{bmatrix} 1/n & \cdots & 1/n \\ 1/n & & 1/n \end{bmatrix}$ $n = \#$ vertices

(random walk independent vertex)

$\|A - J/n\| \leq \lambda$ ($\lambda \ll 1$)

then A is a 'good' expander

(equiv to spectral defn)

Thm: There exists explicit construction of constant degree expanders

— Is there a similar theory of expansion for hypergraphs?

$H = (V, F)$

$\hookrightarrow V$ - vertices

F - set of hyperedges

$F \subseteq \binom{V}{k}$ $\quad (k \geq 3)$

$\Big\}$ k-uniform

— In case of graphs, the gold-standard – complete graph.

Similarly, for k-uniform hypergraphs gold standard – complete k-uniform hypergrap

$/ \; (V, \binom{V}{k})$

— k-uniform hypergraphs (all edges have same arity)

— Multiple Generalization of expansion to hypergraph

- Most generalizations are not equivalent

Today's lecture (high-dimensional expanders)

- a generalization based on random walks / spectral.

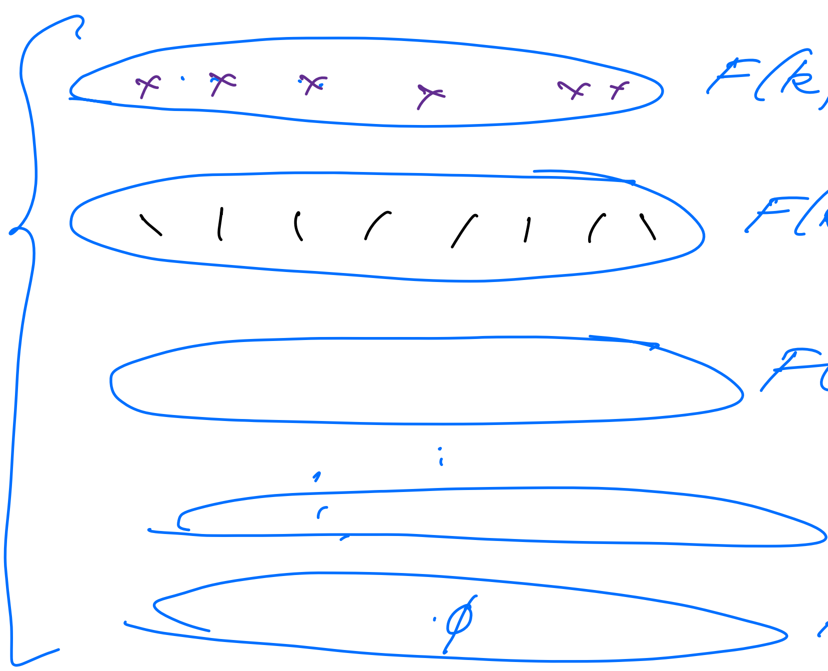
- Simplicial viewpoint of hypergraph

(V, F) $\qquad F \subseteq \binom{V}{k}$

$F = F(k)$ - set of hyperedges of size k.

$\forall\, 0 \le i \le k,$

$$F(i) = \{ S \subseteq \binom{V}{i} \mid \exists\, T \in F,\ S \subseteq T \}$$



$F(k) = F$ - Π_k-uniform

$F(k-1)$ - Π_{k-1} (induced dist).

$F(k-2)$. - Π_{k-2}

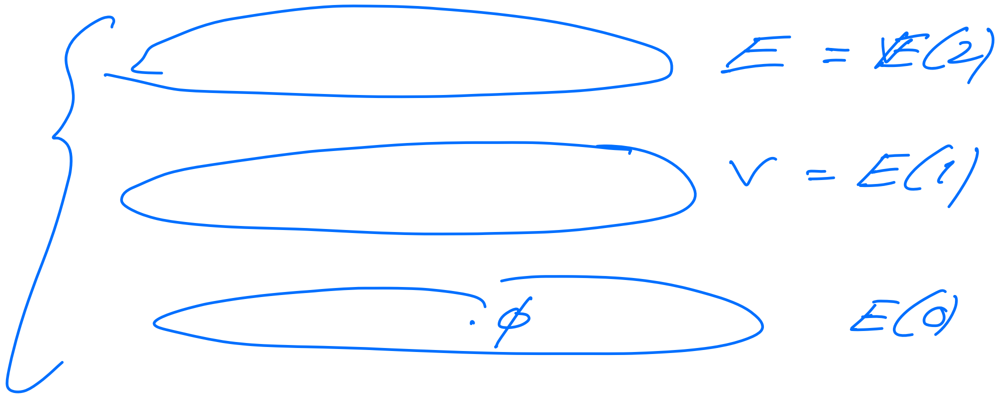
$\vdots$

$F(1)$ - vertices - V.

$F(0)$ - Π_0

ϕ

Graph:

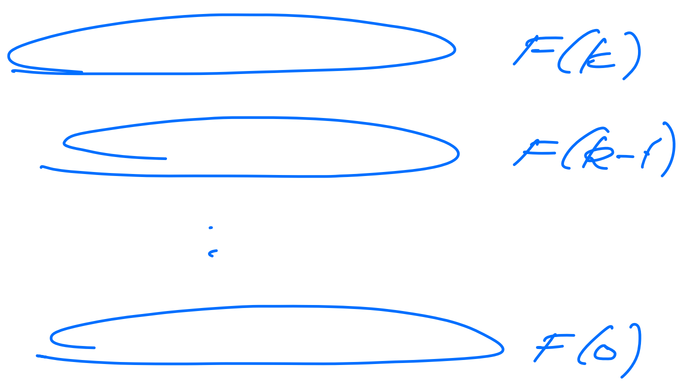


$F = (F(0), F(1), \ldots \; F(k))$

$\hookrightarrow$ down closed family of sets

$F(0) = \{\phi\}$.

Define: suitable notion of expansion.

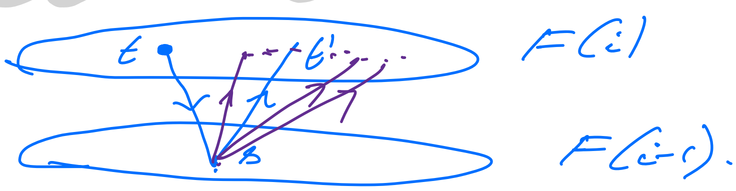


Fix $0 \leq i \leq k$.

$(1 \leq i \leq k-1)$

Two types of walks on layer $F(i)$

Down-Up Walk

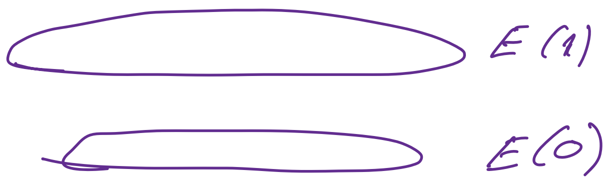


Given $t \in F(i)$

- Let $a \in_u t$
  $s \leftarrow t \setminus \{a\}$
- $t' \supseteq s$.

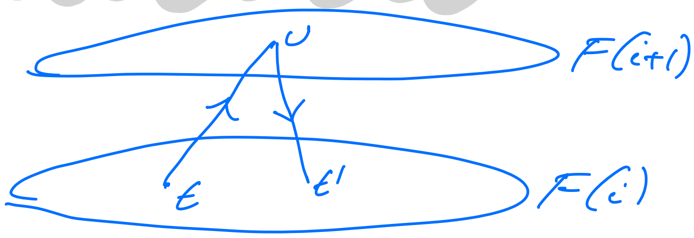
from the dist $\pi_i|_{t'\supseteq s}$

Down-Up Walk on a graph
on layer $E(1) = V$



(regular graph
DU-walk $= J/n$)

Up-Down Walk on layer i

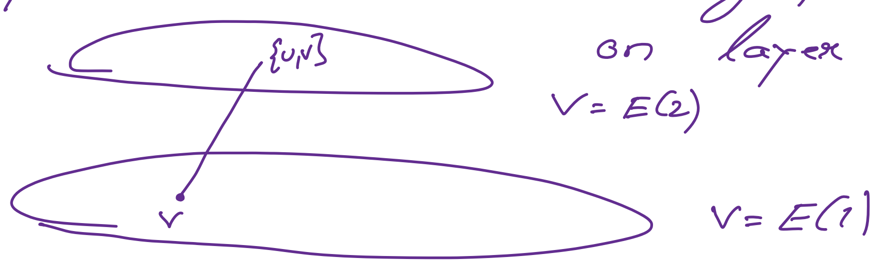


Given $t \in F(i)$

Pick $u \leftarrow \pi_{i+1}|_{u \supseteq t}$

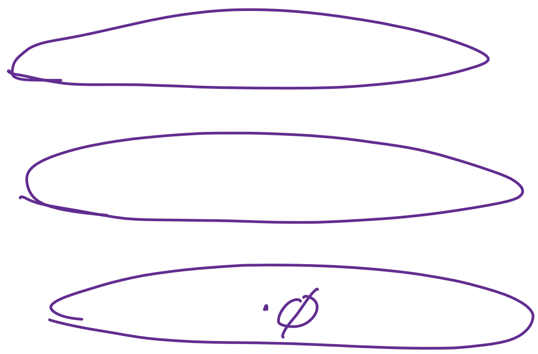
Pick $t' \leftarrow \pi_i|_{t' \subseteq u}$

Up-down walk on a graph
on layer $V = E(1)$



Non-Lazy component of the up-down
walk - usual random
walk on a graph.

Graph: (regular)



$E(2) = E$

$E(1) = V$

$E(0) = \{\emptyset\}$.

$\|A - J/n\| \le \lambda$ $\quad J/n$ – Down-up walk on layer $E(1)$

(ie $\|A - (J-I)/(n-1)\| \le \lambda$) A – non-lazy part of the up-down walk on layer $E(1)$.

Generalize this to hypergraph.

Defn: $(E(k), F(k-1), \dots, F(0))$ – down closed family of sets is a λ-HDX if

$\forall \; 1 \le i \le k-1$

$$\|\text{nl-}P_i^{\wedge} - \text{nl-}P_i^{\vee}\| \le \lambda$$

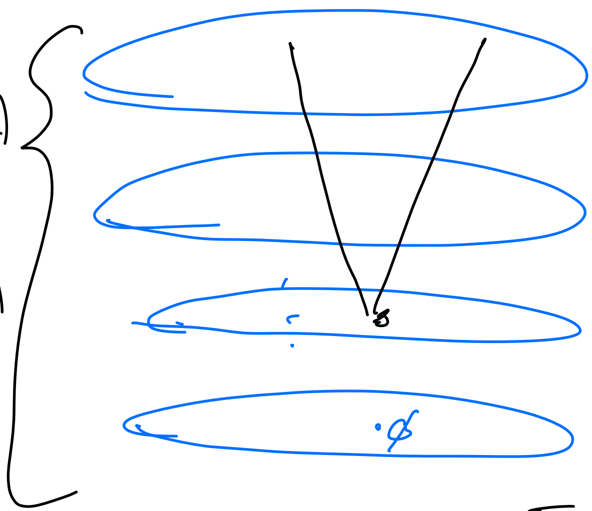
$P_i^{\wedge}$ – up-down walk

$P_i^{\vee}$ – down-up walk.

Another defn of HDX (in terms of links)

Link:

$F = F(0) \cup F(1) \cup \vdots \cup F(k)$



$F(k)$

$F(k-1)$

$F(i)$

$F(0)$

$0 \le i \le k$

$s \in F(i)$

Link

$F_s = \{ t \setminus s \mid t \supseteq s \}$

F_s - down-closed family of sets.

$F = F(0) \cup F(1) \ldots \cup F(k).$

$F(k)$ π_k

$F(2)$ π_2

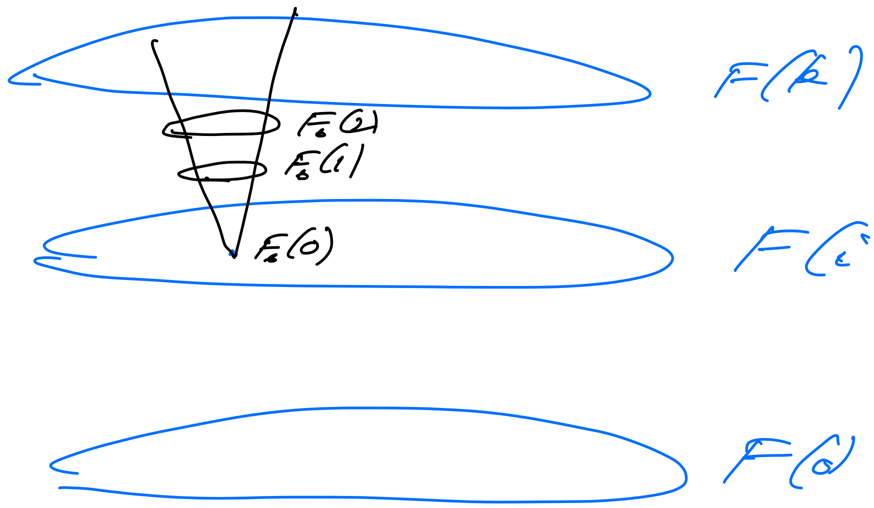
$F(1)$ π_1

$F(0)$ π_0

underlying graph of the hypergraph

Defn: $F = (F(0) \ldots F(k))$ is a λ-link HDX if [Dinur-Kaufman]

for all $0 \le i \le k-2$. & $s \in F(i)$
the underlying graph of F_s
– λ-spectral expander.



Thm: λ-link HDX $\Rightarrow$ λ-HDX
& conversely for constant k
λ-HDX $\Rightarrow$ $k\lambda$-link HDX
(k - arity of sets in the topmost layer).

– Questions:
(1) Do sparse λ-HDX exist?
(2) Are HDX "useful"?

(1) Sparse HDX:

- Lubotsky - Samuels - Vishne. (Ramanujan Complexes)
- Kaufman - Oppenheim (exposition - HS)
- O'Donnell - Pratt

(2) Is this theory of HDX useful?

Up-down / Down-up walks

- help in "counting the number of spanning trees in a graph" (sampling a random spanning tree).

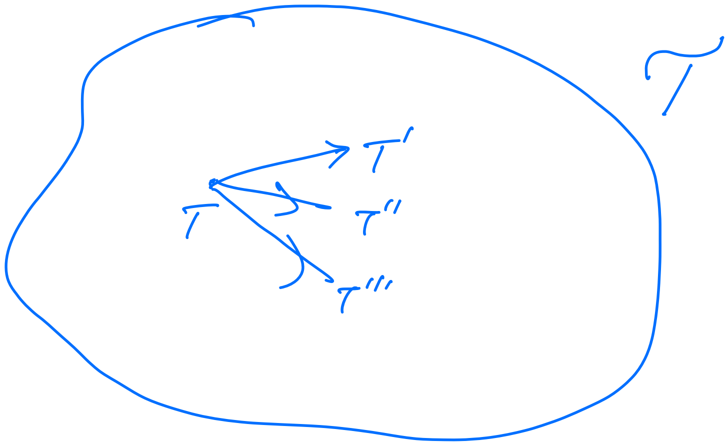
Problem:

Given a graph (undirected) $G = (V, E)$,

$\mathcal{T} = \{ T \subseteq E \mid T$ is a spanning tree of $G \}$.

sample an elt T uniformly $\mathcal{T}$.

Obs: $|\mathcal{T}|$ - may be exponentially large in size of graph G.



Markov chain on $\mathcal{T}$

Random walk on $\mathcal{T}$

Random walk converges very fast to the unif dist.

Glauber Dynamics on $\mathcal{T}$.

$GD_{\mathcal{T}}$: Given a $T \in \mathcal{T}$.

- Pick a $e \in_u T$

  $S := T \setminus \{e\}$

- Pick a random $T' \supseteq S$.

$T' = S \cup \{f\}$
for some $f \in E$
st $S \cup \{f\}$ is a
tree.

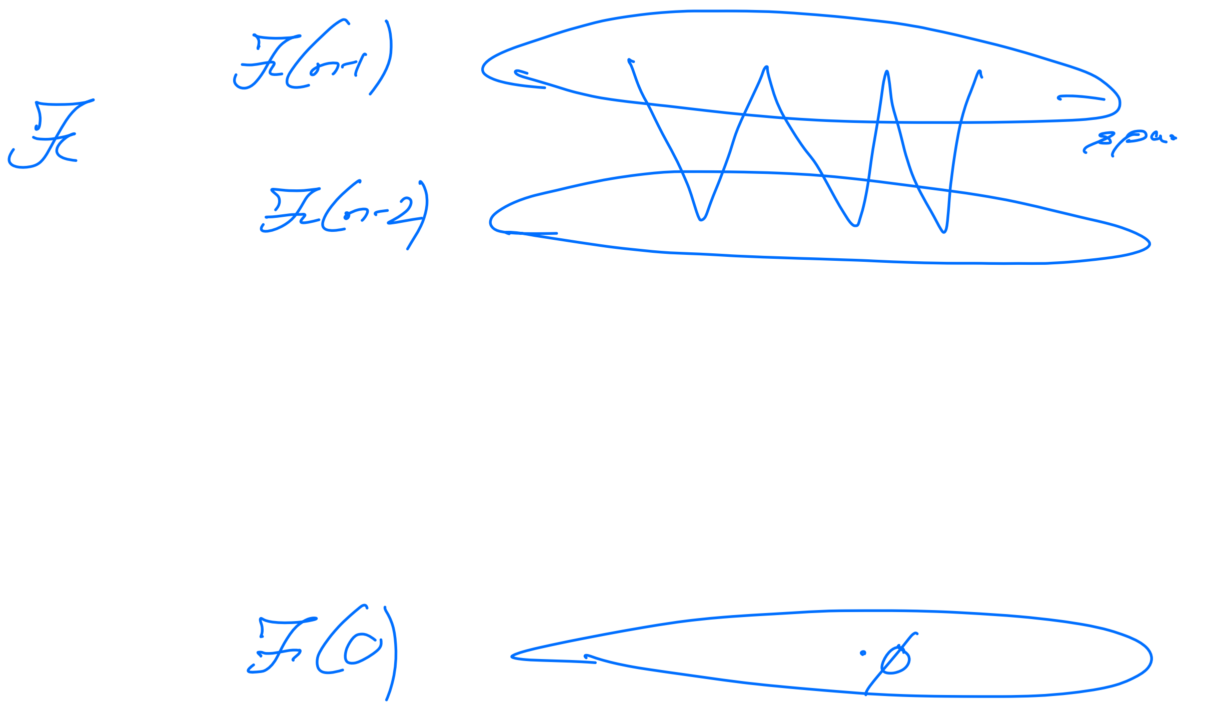
Set of all forests in G

$$\mathcal{F} = \{F \subseteq E \mid F \text{ is a forest}\}$$

- $\mathcal{F}$ - downclosed?

$$\max_{F \in \mathcal{F}} |F| = |V| - 1$$

(assuming G is connected)



GD_T : Down-Up walk on $\mathcal{F}$
on layer $n-1$.

To show: GD_T mixes well

equiv

Down-up walk on layer $n-1$
mixes well.

Next lecture: Prove this by showing
$\mathcal{F}$ is a good HDX.
[Anari-Liu-Charan-Vinzent '19]