Today

- Unique Games Conjecture (UGC)
  - ▹ Vertex Cover

CSS. 330.1 : PCPs

Limits of Approximation Algorithms

Lecture 10 (2023-4-14)

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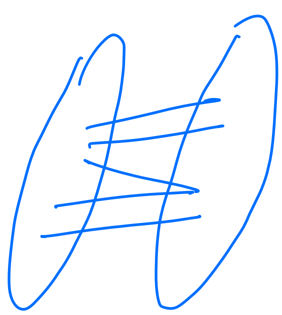
Today

UG-hardness of (2-ε)-approximating Vertex Cover (Khot Regev).

Recall

$\text{gap}_{1-\varepsilon,\delta}\text{-ULC}$



$\Phi = (G = (U, V, E), \Sigma, \psi)$

$\psi = \{ \psi_e : \Sigma \times \Sigma \to \{0,1\} \mid \psi_e \text{ is a matching relation} \}$

$$\text{val}(\Phi) = \max_{\substack{A : U \to \Sigma \\ B : V \to \Sigma}} \Pr_{(u,v)} \left[\psi_{(u,v)}(A(u), B(v)) = 1 \right]$$

YES: $\{ \Phi \mid \text{val}(\Phi) \geq 1 - \varepsilon \}$

NO: $\{ \Phi \mid \text{val}(\Phi) \leq \delta \}$

Khot - Regev Modifications.

Step 1. Move to a non-bipartite.

$$\Phi = (G = (V, E), \Sigma, \varphi)$$

Completeness: $1-\varepsilon \to 1-2\varepsilon$

Soundness: $\delta \to \delta$

Step 2. Strengthen soundness to a list-decoding strngt.

$$NO_\delta = \{\Phi \mid val(\Phi) \leq \delta\}$$

ie, $\forall$ colorings $A: V \to \Sigma$, at most δ-fraction of edges are satisfied.

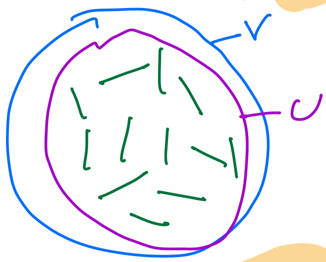
or For every $> \delta$- fraction of edges F & $A: V \to \Sigma$ there is at least one $e \in F$ that is violated by A.

or For every $> \nu$ fraction of edges F & t-coloring $A: V \to \binom{\Sigma}{\leq t}$, there is at least one edge $(u,v) \in F$ s.t

$$(A(u) \times A(v)) \cap \varphi_{(u,v)} = \phi.$$

Step 3: Move from fraction of edges to fraction of vertices.

ie, $gap_{\nu,t,\alpha}$-OLC



YES_γ : $\exists\, U \subseteq V$ & 1-coloring $A: U \to \Sigma$

(i) $|U| \geq (1-\gamma)|V|$

(ii) For all edges $(u,v) \in (U \times U) \cap E$

$(A(u), A(v)) \in \varphi_{(u,v)}$.

$NO_{\gamma,t}$: $\forall\, U \subseteq V$, $|U| \geq \gamma |V|$ & t-colorings

$A: U \to \binom{\Sigma}{\leq t}$.

we have at least an edge $(u,v) \in E \cap (U \times U)$

s.t

$(A(u) \times A(v)) \cap \varphi_{(u,v)} = \phi$.

Khot-Regev : UGC assumption is equivalent to the following.

$\forall\, \gamma \in (0,1)$, $t \in \mathbb{Z}_{>0}$, $\exists\, q$ (size of label set) s.t given an instance Φ of ULC, it is NP-hard to distinguish

$\Phi \in YES_\gamma$ or $\Phi \in NO_{\gamma,t}$.

Reduction to Vertex Cover:

$gap_{\gamma,t,q}$-ULC $\longrightarrow$ Vertex Cover

$\Phi \longmapsto \mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$ (vertex wts)

YES_γ ..... $\mathcal{G}$ will have an $\geq (\frac{1}{2} - \alpha)$-fraction of independent set (ie, G has a vertex cover of size at most $(\frac{1}{2}+\alpha)$)

$NO_{\gamma,t}$ $\cdots\rightarrow$ Every independent set in G is of size at most β

(ie, every VC of G is of size at least $1-\beta$)

$\forall\, \alpha, \beta \in (0,1)$, $\exists\, t, \gamma$ such that above is true.

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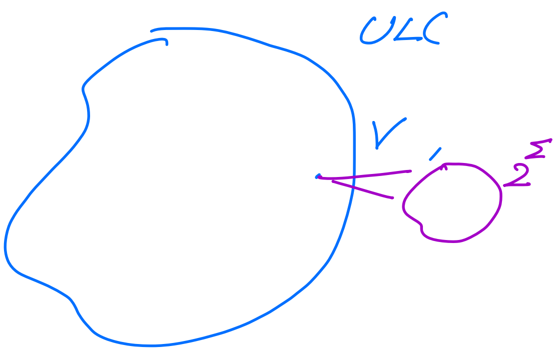
t, γ - will be constants to be specified later.



$((U, E), \Sigma, \psi)$

$|\Sigma| = \kappa$

$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

$\mathcal{V} = V \times 2^{\Sigma}$

$u, v \in V$

$F, G \subseteq \Sigma$

$(u,F) \sim (v,G)$

iff

(i) $(u,v) \in E$

(ii) $(F \times G) \cap \psi_{(u,v)} = \phi$ $\qquad (\frac{1}{2}-\varepsilon)$

$\omega(u,F) = \frac{1}{|V|} \cdot \frac{1}{2^{\kappa}}$; Biased weights $\omega(u,F) = \frac{1}{|V|} (\frac{1}{2}-\varepsilon)^{|F|} (\frac{1}{2}+\varepsilon)^{\kappa - |F|}$

Completeness: $\Phi \in YES_\gamma$.

ie $\exists\ U \subseteq V$, $|U| \geq (1-\gamma)|V|$ & $A: U \to \Sigma$

s.t $\forall\ (u,v) \in (U \times U) \cap E$

$(A(u), A(v)) \in \Psi_{(u,v)}$

$$\mathcal{I} = \{(u,F) \mid u \in U,\ A(u) \in F\}.$$

$\mathcal{I}$ is an independent set.

$$\omega(\mathcal{I}) \geq (1-\gamma)\cdot\frac{1}{2} = \frac{1}{2} - \frac{\gamma}{2}.$$

$$(1-\gamma)\left(\frac{1}{2}-\varepsilon\right) = \frac{1}{2} - \alpha.$$

Claim: $\Phi \in YES_\gamma \Rightarrow \mathcal{G}$ has an independent set of size $\geq \frac{1}{2}(1-\gamma)$.

Soundness:

Claim: There exists a $t = t(\gamma, \varepsilon) \in \mathbb{N}$ s.t $\Phi \in NO_{\gamma,t}$ then $\alpha(\mathcal{G}) \leq 2\gamma$.

(ie largest independent set)

Proof by contradiction.

Assume, there exist $\mathcal{I} \subseteq \mathcal{V}$ s.t $\omega(\mathcal{I}) \geq 2\gamma$.

I_v: - restriction of I to cloud $v \times 2^\Sigma$

ie, $\mathcal{I}_v \triangleq (v \times 2^{\Sigma}) \cap \mathcal{I}$

$U \triangleq \{v \in V \mid \omega(\mathcal{I}_v) \geq \nu\}$.

$|U| \geq \nu |V|$ by averaging.

Goal: Find a t-coloring for U.

$(u,F), (v,G) \in \mathcal{I}$.
then either

(i) $(u,v) \notin E$

or

(ii) $(u,v) \in E$ & $(F \times G) \cap \Psi_{(u,v)} \neq \emptyset$

For any $F' \supseteq F$, $G' \supseteq G$, there is no edge between (u,F') & (v,G')

If $\mathcal{I}$ is maximal, then each $\mathcal{I}_v$ is monotone (up-closed).

Digression into Extremal Combinatorics:

Friedgut's theorem gives a suff't condn for a fn to be approximated by a junta (namely, low influence)

$\mu_p^{\otimes n}$ $\quad p \in (0,1)$ $\quad f: \{0,1\}^n \to \{0,1\}$

Influence: $\mathrm{Inf}_i^p(f) = \mathbb{E}_{x_{-i} \sim \mu_p^{\otimes n-1}} \left[\mathrm{Var}_{x_i \sim \mu_p} \left(f(x_1 \ldots x_{i-1}, x_i, x_{i+1} \ldots, x_n) \right) \right]$

$$= p(1-p) \Pr_{x \sim \mu_p^{\otimes n}} \left[f(x) \neq f(x+e_i) \right]$$

Total Influence: $I^p(f) = \sum \mathrm{Inf}_i^p(f)$

Friedgut's Theorem: Let $p \in (1/3, 2/3)$ & $\delta \in (0,1)$, $f: \{0,1\}^n \to \{0,1\}$ & $I^p(f) \leq k$, then there exist a junta $g: \{0,1\}^n \to \{0,1\}$ depending on at most $\exp(O(k/\delta))$ variables s.t.

$$\Pr_{x \sim \mu_p^{\otimes n}} [f(x) \neq g(x)] \leq \delta.$$

Monotone functions:

$\mu_p(f) = \mu_p(x \mid f(x) = 1)$

For non-constant monotone fn.

μ_p goes from 0 to 1 as p goes from 0 to 1

Russo's Lemma: $\dfrac{d\mu_p(f)}{dp} = \dfrac{I^p(f)}{p(1-p)}$

Dinur-Safra Lemma for Monotone functions:
Let $p \in (1/3, 2/3)$. Fix $\delta \in (0,1)$, $f: \{0,1\}^n \to \{0,1\}$ then there exist a $q \in (p, p+\varepsilon/2)$ and a junta g depending on at most $\exp(O(1/\delta\varepsilon))$ variables s.t

$$\Pr_{x \sim \mu_q^{\otimes n}} [f(x) \neq g(x)] \leq \delta.$$

Applying above lemma to each of I_v, $v \in U$ we have that there exist

- $q_v \in \left(\frac{1}{2}-\varepsilon, \frac{1}{2}-\frac{\varepsilon}{2}\right)$
- $f_v: \{0,1\}^n \to \{0,1\}$ is a $t(\varepsilon,\delta) = \exp\left(O\left(\frac{1}{\delta\varepsilon}\right)\right)$ junta depending on $G_v \subseteq [n]$ vars.
- $\Pr_{x \sim \mu_{q_v}^{\otimes n}} [f_v(x) \neq I_v(x)] \leq \delta.$

Define a t-coloring.

$$A: U \to \binom{\Sigma}{\leq t}$$

$$v \mapsto G_v.$$

We would like to show that for all edges $(u,v) \in (U \times U) \cap E$, we have.

$$(G_u \times G_v) \cap \psi_{(u,v)} \neq \phi$$

Crs A is a valid $t$coloring of U & hence

Φ is not a $NO_{\gamma,\epsilon}$-instance).

Fix one such edge (u,v).

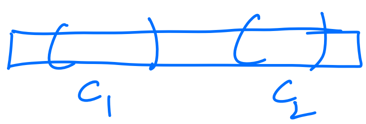
$$\psi_{(u,v)} \quad \begin{array}{ccccc} u & q_u & I_u & f_u & C_u \\ v & q_v & I_v & f_v & C_v \end{array}$$

& suppose $(C_u \times C_v) \cap \psi_{(u,v)} = \emptyset$

To simplify notation assume $\psi_{u,v}$ is the identity permutation

$$\text{Identity perm} \quad \begin{array}{ccccc} u_1 & q_1 & I_1 & f_1 & C_1 \\ u_2 & q_2 & I_2 & f_2 & C_2 \end{array}$$

$C_1 \cap C_2 = \emptyset$



n co-ordinates

Suffices to demonstrate $F_1 \in I_1$ & $F_2 \in I_2$

s.t. $(u_1, F_1) \sim_{G} (u_2, F_2)$.

Sample F_1 & $F_2 \subseteq [n]$ as follows.

For each $\sigma \in \Sigma$, put σ in F_1 w/ prob q_1
put σ in F_2 w/ prob q_2
put in neither w/ prob $1-(q_1+q_2)$

Any (F_1, F_2) sampled in the above manner are disjoint.

$(u_1, F_1) \sim_{G} (u_2, F_2)$ w/ probability 1

All that we need to show are

$F_1 \in I_1$ & $F_2 \in I_2$

(i.e. $\Pr[F_1 \in I_1 \wedge F_2 \in I_2] > 0$)

What about

$$\Pr_{F_1,F_2}[f_1(F_1) = 1 \wedge f_2(F_2) = 1]$$

$$= \Pr_{F_1}[f_1(F_1) = 1] \cdot \Pr_{F_2}[f_2(F_2) = 1]$$

(since $C_1 \cap C_2 = \emptyset$ & f_i is C_i-junta)

$$= \mu_{q_1}(f_1) \cdot \mu_{q_2}(f_2)$$

$$\mu_{q_i}(f_i) \geq \mu_{q_i}(I_i) - \delta$$

$$\geq \mu_p(I_i) - \delta \qquad (\text{where } p = \tfrac{1}{2} - \varepsilon)$$

$$\geq \nu - \delta$$

$$\geq \nu/2 \qquad \text{if } \delta \leq \nu/2.$$

Hence $\Pr_{F_1,F_2}[f_1(F_1) = 1 \wedge f_2(F_2) = 1] \geq \nu^2/4.$

$$\Pr_{F_1,F_2}[F_1 \in I_1 \wedge F_2 \in I_2]$$

$$\geq \Pr_{F_1,F_2}[f_1(F_1) = 1 \wedge f_2(F_2) = 1]$$

$$- \Pr[f_1(F_1) = 1 \wedge I_1(F_1) \neq 1]$$

$$- \Pr[f_1(F_2) = 1 \wedge I_2(F_2) \neq 1]$$

$\geq \nu^2/4 - 2\delta$

$\geq \nu^2/8$ if $\delta \leq \nu^2/16$

Set $\delta = \nu^2/16$; $t = t(\varepsilon, \delta)$.

$\Pr[F_1 \in I_1 \wedge F_2 \in I_2] > 0$.

Hence I_1 I_2 are not an ind set

$\boxtimes$

Part II:

Improved PCP constructions
(PCPs w/ very low soundness error bypassing the use of ‖ repetition theorem)

PCP Theorem (Low-Degree Test) $\longrightarrow$ Parallel Repetition Thm

$gap_{1,0.99}$-LC hardness

$gap_{1,\delta}$-LC hardness

Credn - $n^{O(\log 1/\delta)}$ time)

Qn: Can we obtain $gap_{1,\delta}$-LC hardness for $\delta = o(1)$?

Application: [Feige] $gap_{1,1/\log n}$-LC $\rightarrow$ $(\ln n)(1-\varepsilon)$-approx SET Cover.

Two potential approaches:

(1) Improve LDT to get PCPs w/ very low soundness error bypassing || repetition thm

(2) "Derandomize" the || repetition thm.

[Feige-Kilian] - Strong negative results. (rule out inverse exponential decay)

(1) (a) Raz-Safra & Arora-Sudan (Improved LDT analyses)

(b) Alphabet Reduction Technique (Moshkovitz-Raz) Dinur-Harsha.

(2) Inverse Polynomial Derandomized || repetition thm

Open (in full generality)

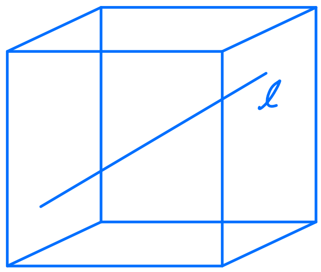
Dinur-Meir: (i) Can do this for "linear" games

(ii) Blowup alphabet to get linear structure.

(iii) Apply alphabet redn technique from 1(b).

Low-Degree Test

$\mathbb{F}$ - finite field; m - dimension; d - degree



$f: \mathbb{F}^m \to \mathbb{F}$

$\mathcal{A}: \{\text{lines}\} \to P(1,d)$

$LDT^{f,\mathcal{A}}(\mathbb{F}, m, d)$

1. Pick $x \in_R \mathbb{F}^m$
2. Pick $\ell \in_R \{\text{lines} \,/\, \ell \ni x\}$
3. Query f on x &
   $\mathcal{A}$ on ℓ
4. Accept if $f(x) = \mathcal{A}(\ell)(x)$

Friedl-Sudan LDT Soundness Theorem. $\exists\, C$ $\forall \varepsilon \in (0,1)$, $\mathbb{F}$ - finite field, m, d s.t.

$$|\mathbb{F}| \geq \max\{Cd, O(\tfrac{1}{\varepsilon^2})\},$$ then the following holds

$\forall f, \mathcal{A}$:

$$\Pr[(f,\mathcal{A}) \text{ passes } LDT] \geq 1-\delta \text{ for } \delta < \tfrac{1}{8}\cdot\varepsilon$$

$$\Downarrow$$

$$\delta(f, P(m,d)) \leq 4\delta.$$

What is $\Pr[\text{LDT accepts}] \geq \varepsilon$

$\Downarrow$?

$\exists\ P \in \mathcal{P}(m,d),\quad agr(f, P) \geq \varepsilon'$. for some $\varepsilon' = \varepsilon'(m, d, \mathbb{F})$

Arora-Sudan: Improved LDT analysis
$\forall$ fields $\mathbb{F}$, dim m, deg g. there exists

$\varepsilon_0 = m\, \mathrm{poly}\left(d, \frac{1}{|\mathbb{F}|}\right)$ s.t.

$$\mathbb{E}_{\ell}\left[agr(f|_{\ell}, \mathcal{P}(1,d))\right] \geq \varepsilon \Rightarrow agr(f, \mathcal{P}(m,d)) \geq \varepsilon - \varepsilon_0.$$

Raz-Safra: Variant of Lines-point LDT

Plane-point test: random line $\longrightarrow$ random plane.

Raz-Safra LDT analyses
$\forall$ fields $\mathbb{F}$, dim $m \geq 2$, deg d, there exist

$\varepsilon_0 = m \cdot \mathrm{poly}\left(\frac{d}{|\mathbb{F}|}\right)$, s.t.

$$\mathbb{E}_{s}\left[agr(f|_{s}, \mathcal{P}(2,d))\right] \geq \varepsilon \Rightarrow agr(f, \mathcal{P}(m,d)) \geq \varepsilon - \varepsilon_0.$$

Comparison between

	Arora-Sudan	Raz-Safra
1	✓ line-point	plane-point
2	$\varepsilon_0 = m \cdot poly(d, \frac{1}{\lvert\mathbb{F}\rvert})$	✓ $\varepsilon_0 = m \cdot poly(\frac{d}{\lvert\mathbb{F}\rvert})$
3	Algebraic	Combinatorial.