Today

- Raz-Safra LDT
- PCPs from LDT
- Alphabet Reduction.

CSS. 330.1 : PCPs
Limits of Approximation Algorithms

Lecture 11 (2023-4-21)

Instructor: Prahladh Harsha

Recap from last time:

Plane-Point test [Raz-Safra].

$\forall$ fields $\mathbb{F}$, dim $m \geq 2$, deg d., there exist

$$\varepsilon_0 = m \cdot \text{poly}\left(\frac{d}{|\mathbb{F}|}\right), \text{ s.t}$$

$$\mathbb{E}_{\substack{s \\ \text{2 plane}}}\left[\text{agr}\left(f|_s, P(2,d)\right)\right] \geq \varepsilon \Rightarrow \text{agr}(f, P(m,d)) \geq \varepsilon - \varepsilon_0.$$

Plan for today:

1. High level sketch of Raz-Safra Analysis
2. Constructs PCPs from LDT
3. Alphabet Reduction.

Raz-Safra Low-Degree Test:

Lecture: $m = 3$

Raz-Safra (planes in a cube)

$\exists\ \varepsilon_0 = \mathrm{poly}\left(\frac{d}{|\mathbb{F}|}\right)$ s.t. $\forall\ f: \mathbb{F}^3 \to \mathbb{F}$

$$\mathrm{agr}(f, P(3,d)) \geq \mathbb{E}_{s\text{-plane}}\left[\mathrm{agr}(f|_s, P(2,d))\right] - \varepsilon_0$$

The same proof generalizes to the following:

Raz-Safra ((m-1)-hyperplanes in m dimensions)

$\exists\ \varepsilon_0 = \mathrm{poly}\left(\frac{d}{|\mathbb{F}|}\right)$ s.t. $\forall\ f: \mathbb{F}^3 \to \mathbb{F}$

$$\mathrm{agr}(f, P(m,d)) \geq \mathbb{E}_{\substack{s\text{-}(m-1)\\ \text{dim hyperplane.}}}\left[\mathrm{agr}(f|_s, P(m-1,d))\right] - \varepsilon_0$$

Induction:

$$\mathrm{agr}(f, P(m,d)) \geq \mathbb{E}_{s\text{-plane}}\left[\mathrm{agr}(f|_s, P(2,d))\right] - (m-2)\varepsilon_0$$

Planes in a Cube:

$f: \mathbb{F}^3 \to \mathbb{F}$ $\quad$: $\quad$ $F: \{\text{planes}\} \to P(2,d)$

$s \mapsto$ best-fit polynomial for that plane (break ties arbitrarily)

$$\delta \triangleq \Pr_{s,x}\left[f(x) = F(s)(x)\right]$$

$$\Pr_{s,s'}\left[F(s)|_{s\cap s'} \equiv F(s')|_{s\cap s'}\right] \triangleq \varepsilon.$$

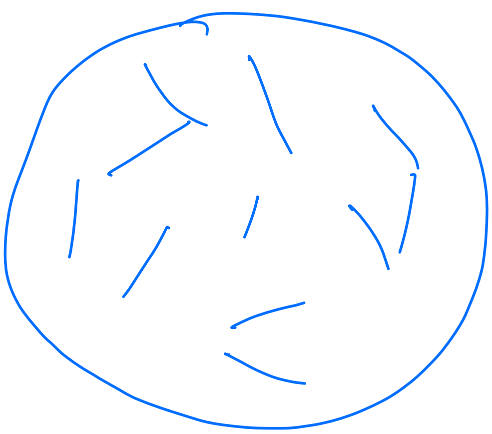
$$\Pr_{x,s,s'}\left[F(s)|_x = f(x) = F(s')|_x\right]$$

$$= \mathop{\mathbb{E}}_x\left[\left(\Pr_{s\ni x}\left[F(s)(x) = f(x)\right]\right)^2\right]$$

$$\geq \left(\mathop{\mathbb{E}}_x\left[\Pr_{s\ni x}\left[F(s)(x) = f(x)\right]\right]\right)^2$$

$$= \delta^2$$

Hence, $\underbrace{\Pr_{s,s'}\left[F(s)|_{s\cap s'} = F(s')|_{s\cap s'}\right]}_{\text{consistency}} \geq \delta^2 - \frac{d}{q} - \frac{1}{q}$



G

$V =$ Planes

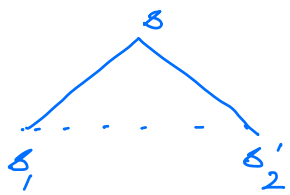
$E =$ Planes that are parallel or consistent w/ each other.

(Consistent: If two planes agree on their intersection)

Dense graph (at least $\delta^2 |V|^2$)

Want to prove: Large clique in the graph

What prevents cliques:



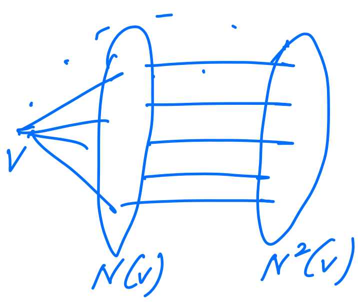
Lemma: Suppose $(s_1, s_2) \notin E$ in the consistency graph

$$\Pr_s\left[(s, s_1) \in E \ \& \ (s, s_2) \in E\right] \le \frac{d+1}{q}.$$

$$\varepsilon \triangleq \frac{d+1}{q}$$

Pruning Process:

1. Do the following
   (a) If there exists a vertex v of degree $\le 2\sqrt{\varepsilon}\,|V|$, then remove all edges out of v.
   (b) Otherwise remove all edges between $N(v)$ & $N^2(v)$.



$$|N(v)| \ge 2\sqrt{\varepsilon}\,|V|$$

End product (post pruning)

G – union of cliques.

At least one clique must be large

Interpolate the large clique to obtain a global polynomial.

Raz-Safra:

$\exists\ \varepsilon_0 = m\,\mathrm{poly}\left(\frac{d}{|\mathbb{F}|}\right)$ s.t. $\forall f: \mathbb{F}^m \to \mathbb{F}$

$$\mathop{\mathbb{E}}_{s\text{-plane}}\left[\mathrm{agr}(f|_s, P(2,d))\right] \geq \varepsilon \Rightarrow \mathrm{agr}(f, P(m,d)) \geq \varepsilon - \varepsilon_0.$$

Equivalent formulation

List-decoding stmt:

$\exists\ \delta_0 = m\,\mathrm{poly}\left(\frac{d}{|\mathbb{F}|}\right)$, s.t. $\forall\ \delta \geq \delta_0$

given any $f: \mathbb{F}^m \to \mathbb{F}$, $\exists\ Q_1, \ldots Q_L \in P(m,d)$

where $L = O(1/\delta)$, $\forall\ F: \{\text{planes}\} \to P(2,d)$

$$\Pr_{x,s}\left[f(x) = F(s)(x) \wedge \nexists i \in [L],\ Q_i|_s \equiv F(s)\right] \leq \delta.$$

Step 2 of today's Plan:

Construct a PCP from this LDT.

Simpler Question:

Is $f: \mathbb{F}^m \to \mathbb{F}$ close to a low-degree poly

$Q: \mathbb{F}^m \to \mathbb{F}$ such that $Q|_{H^m} \equiv 0$.

Claim: $Q|_{H^m} \equiv 0$ iff $\exists\ Q_1 \ldots Q_m$ s.t.

$$Q(x) = \sum_{i=1}^{m} g_H(x_i)\, Q_i(x)$$

where $g_H(z) = \prod_{h \in H} (z - h)$

Proof: $\bar{q} : \mathbb{F}^m \to \mathbb{F}^m$

$\bar{q}(x_1 .. x_m) = (q_1(x), q_2(x) \dots , q_m(x))$

where each q_i – Q_i honest proof).

Zero-on-SubCube Test:

Proof: $\bar{q} : \mathbb{F}^m \to \mathbb{F}^{m+1}$

Test. 1. Pick a random plane s

2. Query $\bar{q}$ on s.

3. Check that $\bar{q}_{i}|_s$ is low-degree and reject otherwise

4. Accept $q_0(x) = \sum q_4(x_i) \, q_i(x)$, $\forall x \in s$.

$\forall$ functions $\bar{q} : \mathbb{F}^m \to \mathbb{F}^{m+1}$

$\exists \, \bar{Q}_1 \dots \bar{Q}_L : \mathbb{F}^m \to \mathbb{F}^{m+1}$ (honest proofs)

$\Pr_s [\, \bar{q}$ passes the Zero-on-Subcube test $\wedge \; \exists i \in [L], \; \bar{q}|_s \neq Q_i|_s \,] \le \delta + \frac{Ld}{q}$

Can do same thing for arithmetization:

Given 3SAT formula Φ.

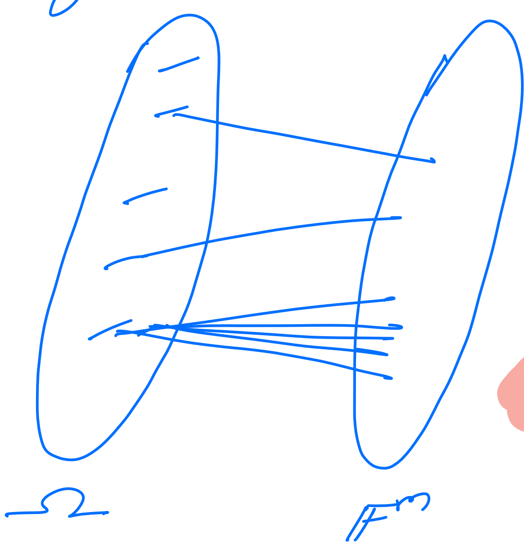
$\exists$ test. Π which queries a 3-dim object Ω in $\mathbb{F}^m$

$\forall \bar{q} : \mathbb{F}^m \to \mathbb{F}^{O(m)}, \ \forall \delta$

$\exists \ \bar{Q}_1, \dots, \bar{Q}_L : \mathbb{F}^m \to \mathbb{F}^{O(m)}$ (honest proofs)

$\Pr_{\Omega}\left[\bar{q}|_{\Omega} \text{ passes test } \Pi \wedge \nexists i: \bar{Q}_i|_{\Omega} \equiv \bar{q}|_{\Omega}\right] \le \delta.$

Converting to Label Cover Problem



$\bar{q} : \mathbb{F}^m \to \mathbb{F}^{O(m)}$

$\text{gap}_{1,\delta}$-LC is δ

Qn: Are we done?

No, large alphabet size.