CSS.330.1 : PCPs

Limits of Approximation Algorithms

Lecture 12 (2023-5-3)

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**Today**

- 2-to-2 games Thm (Introduction)

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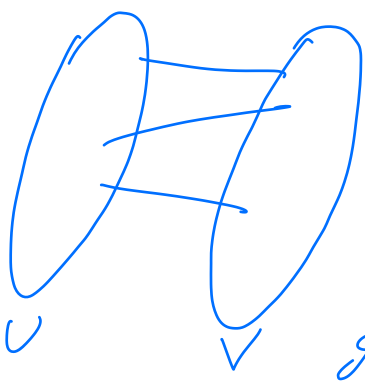
Recall Unique Games Problem.

Label Cover $\Phi = (G = (U, V, E), \Sigma, \Pi)$




$\Pi = \{\pi_e : \Sigma \times \Sigma \to \{0,1\} \mid e \in E\}$

$val(\Phi) = \max_{A : U \cup V \to \Sigma} \Pr_{e=(u,v)} [\pi_e(A(u), A(v)) = 1]$

$0 \le s < c \le 1$

$gap_{c,s}\text{-}LC$ : YES $= \{\Phi \mid val(\Phi) \ge c\}$

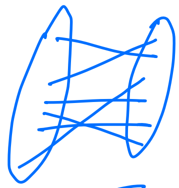
NO $= \{\Phi \mid val(\Phi) \le s\}$

__PCP Thm + Parallel Rep thm__ : $\forall \varepsilon \in (0,1), \exists \Sigma, (|\Sigma| = poly(\frac{1}{\varepsilon}))$ s.t $gap_{1,\varepsilon}\text{-}LC(\Sigma)$ is NP-hard. (under $O(n^{\log(\frac{1}{\varepsilon})})$-time reductions)

Khot suggested a strengthening of above theorem

ULC instances. $\pi_e : \Sigma \times \Sigma \to \{0,1\}$ is a matching.

e



$\Sigma$ $\Sigma$

**UGC**: $\forall \varepsilon, \exists \Sigma$ s.t. $gap_{1-\varepsilon,\varepsilon}$-$ULC(\Sigma)$ is NP-hard

Efforts to refute UGC:

Arora-Barak-Steurer gave an alg that runs in time $2^{n^{\varepsilon'}}$ s.t.

(1) $(1-\varepsilon)$-satisfiable ULC instances, it found an $\frac{3}{4}$-satisfying assignment

(2) On $\frac{1}{2}$-satisfiable ULC instance, it found a $\varepsilon$-satisfying assignment.

$(\varepsilon' \to 0$ as $\varepsilon \to 0)$

Moshkovitz-Raz Theorem

$\forall \varepsilon, \exists \Sigma \cdot (|\Sigma| = \exp(\frac{1}{\varepsilon}))$ s.t.

$gap_{1,\varepsilon}$-LC is NP-hard under $O(n^{1+o(1)} \cdot poly(\frac{1}{\varepsilon}))$ time-reductions.

Inability to find a distribution of hard instances for UG.

— Khot-Minzer-Safra [KMS1] '17

Dinur-Khot-Kindler-Minzer-Safra [DKKMS1] '18

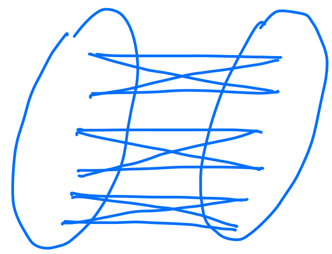
DKKMS2 '18

KMS2 '18

A variant of UG is actually NP-hard.

2-to-2 Games.

$\pi_e : \Sigma \times \Sigma \to \{0,1\}$




Graph of each constraint $\pi_e$ is a union of even cycles.

$\ell \in \mathbb{Z}_{>0}$

2-to-2-games $[\mathbb{F}_2^\ell]$

$\Phi$ : $n$ vars $x_1 \ldots x_n \in \mathbb{F}_2^\ell$

$m$ constraints $C_1 \ldots C_m$.

where each constraint $C$ is of the form

$T_1 x_i \oplus T_2 x_j \in \{b_1, b_2\}$

where $T_1, T_2$ are invertible $\mathbb{F}^{\ell \times \ell}$-matrices

$b_1, b_2 \in \mathbb{F}_2^\ell$.

$C = (i, j, T_1, T_2, b_1, b_2)$.

$\text{gap}_{c,s}$-2-to-2-games $[\mathbb{F}_2^\ell]$ - defined.

Thm [KMS1, DKKMS1, DKKMS2, KMS2]

$\forall \varepsilon \in (0,1)$, $\exists\ \ell = \ell(\varepsilon)$ s.t. $\text{gap}_{1-\varepsilon, \varepsilon}$-2-to-2$[\mathbb{F}_2^\ell]$ is NP-hard under $n^{\text{poly}(1/\varepsilon)}$-time reductions

Cor: $\forall \varepsilon \in (0,1)$, $\exists \Sigma$, $gap_{\frac{1}{2}-\varepsilon,\varepsilon}$-ULC is NP-hard.

## Consequences for inapproximability.

1. Vertex Cover: $\forall \varepsilon \in (0,1)$, $\sqrt{2}(1-\varepsilon)$-approx of VC is NP-hard.

2. Graph Coloring:

Dinur-Mossel-Regev:

$\forall \varepsilon \in (0,1)$, NP-Hard to distinguish
- graphs which are at most 4 colorable
- graphs whose largest ind set is $\varepsilon$ (hence, $\chi(G) \geq 1/\varepsilon$)

## Proof of the 2-to-2 Games Theorem

Starting Point $gap_{1-\varepsilon,\frac{3}{4}}$-3LIN2.

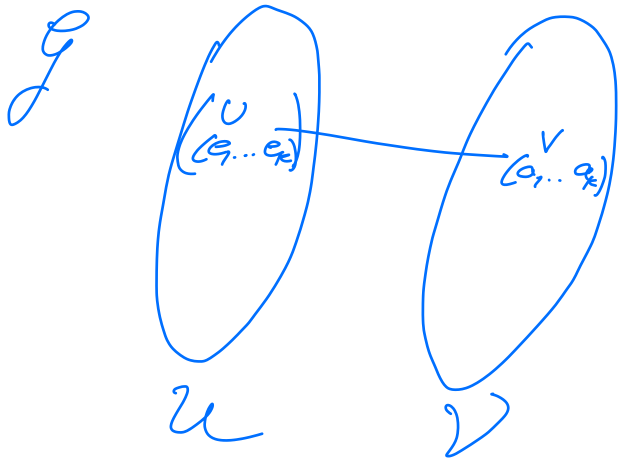
$$\left\{ \begin{array}{l} x_i + x_j + x_k = b. \\ \quad \vdots \\ \quad \vdots \end{array} \right\} \begin{array}{l} n\text{- vars} \\ m\text{- equations} \end{array}$$

Outer PCP from $gap_{1-\varepsilon,3/4}$-3LIN2. instance $\psi$

- $k$ (s.t. $k\varepsilon \ll 1$), $\beta$
- Pick $k$ random equations $e_1 \ldots e_k$

$\Phi$

$\mathcal{G}$




For each $i \in [k]$

$$a_i \leftarrow \begin{cases} e_i & \text{w/ prob } 1-\beta \\ \text{random var in } e_i & \text{w/ prob } \beta \end{cases}$$

$U =$ vars in $(e_1 \ldots e_k)$

$V =$ vars in $(a_1 \ldots a_k)$

$|U| \simeq 3k \quad ; \quad |V| \approx 3k - 2_\beta k.$

$(U, \bar{e})$ —— $V$ : edge in $\mathcal{G}$.

$\alpha \in \mathbb{F}_2^U \qquad \beta \in \mathbb{F}_2^V$ – labels

Consistency Test: (1) $\alpha$ satisfies all lin eqns in $\bar{e}$

(2) $\beta$ is a projection of $\alpha$.

$\text{gap}_{1-\varepsilon, \frac{3}{2}}\text{-}3LIN \longrightarrow \text{gap-}LC$

$\psi \longmapsto \Phi$

**Proposition**: The above is a redn from $\text{gap}_{1-\varepsilon, \frac{3}{4}}\text{-}3LIN2$ to $\text{gap}_{1-k\varepsilon, 2^{-\Omega(\beta k)}}\text{-}LC$

## Grassmann Code:

$x \in \mathbb{F}_2^k \qquad f_x : \mathbb{F}_2^k \to \mathbb{F}_2$

$z \mapsto \langle x, z \rangle$

$\forall L, \; L \subseteq \mathbb{F}_2^k$ subspace $\dim(L) = \ell$

$f_x|_L$

$F_x : \begin{bmatrix} k \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^\ell$ (Here $\begin{bmatrix} k \\ \ell \end{bmatrix}_2$ refers to all $\ell$-dim subspaces of $\mathbb{F}_2^k$)

For each $u \in \mathcal{U}$ $\quad F_u : \begin{bmatrix} 3k \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^\ell$

$v \in \mathcal{V}, \quad F_v : \begin{bmatrix} N' \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^\ell$.

**Qn:** Given $F : \begin{bmatrix} k \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^\ell$, test if this is an encoding of an $x \in \mathbb{F}_2^k$ (or equivalently is it a restriction of a linear fn)

$\text{Test}_{Gr,t}$ : Input: $F : \begin{bmatrix} k \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^\ell$ (oracle)

1. Pick $L, L' \in \begin{bmatrix} k \\ \ell \end{bmatrix}_2$ s.t. $\dim(L \cap L') = t$
2. Accept if $F[L]|_{L \cap L'} \cong F[L']|_{L \cap L'}$

Not to hard to prove the:

Thm: Suppose $\Pr[\text{Test}^F_{Gr,t} \text{ acc}] \geq \delta$, then there exists a global linear function $f$ s.t

$$\Pr_L [F[L] \equiv f|_L] \geq \Omega(\delta^3)$$

provided $1 \leq t \leq \ell/4$ & $\delta \geq \frac{6}{2^{t/4}}$.

This yields a $2^{\ell-t}$-to-$2^{\ell-t}$-test.

We want $t = \ell - 1$, but then the theorem does not hold.

$\boxed{\begin{array}{l} \text{Test}_{Gr,lin}\text{: Input: } F : \begin{bmatrix} k \\ \ell \end{bmatrix}_2 \to \mathbb{F}_2^{\ell} \text{ (oracle)} \\ \quad 1.\ \text{Pick } L, L' \in \begin{bmatrix} k \\ \ell \end{bmatrix}_2 \text{ s.t } \dim(L \cap L') = \ell - 1 \\ \quad 2.\ \text{Accept if } F[L]|_{L\cap L'} \overset{?}{=} F[L']|_{L \cap L'} \end{array}}$

Qn: $\forall \delta, \exists \varepsilon$. $\forall, \ell, k$ s.t if $\Pr[\text{Test}^F_{Gr,lin} \text{ acc}] > \delta$ then $\exists$ global linear fn $f$

$$\Pr_L [F[L] = f|_L] \geq \varepsilon.$$

Consider $Gr(k, \ell)$ : Vertices are $\begin{bmatrix} k \\ \ell \end{bmatrix}_2$.

$(L, L')$ is an edge if $\dim(L \cap L') = \ell - 1$

$Gr(k, \ell, t)$ : $L \sim L'$ if $\dim(L \cap L') = t$

Fix a direction $v$

$S_v = \{L \mid v \in L\}$.

Obs: $S_v$ is a non-expanding set

ie, $\varphi(S_v) = \dfrac{E(S_v, \overline{S_v})}{d\,|S_v|} \approx \dfrac{1}{2}$

Any collection of $S_v$'s such

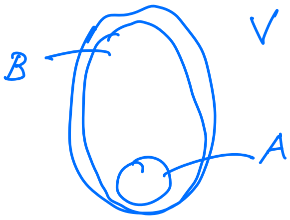
(1) pairwise disjoint (almost).

(2) union is a $\delta$-fraction of $\begin{bmatrix} k \\ \ell \end{bmatrix}_2$.

will serve as a counterexample to the conjecture.

$A, B$ are subspace of dim $a$ & co-dim $b$

$A \subseteq B$ st $a + b \le r$.




$S(A,B) = \{L \mid A \subseteq L \subseteq B\}$.

**Thm** [KMS2]: $\forall \delta$, $\exists \varepsilon$ & integer $r$ such that
$\forall$ sufficiently large $l$ & $k \geq l$.
let $S \subseteq V(Gr(k,l))$ st
$$\varphi(S) \leq \delta$$
then $\exists$ subspaces $A \subseteq B$ st
$$\dim(A) + \text{co-dim}(B) \leq r \quad \text{st}$$
$$\frac{|S \cap S(A,B)|}{|S(A,B)|} \geq \varepsilon.$$

**Thm** [DKKMS2]
$\forall \delta$, $\exists \varepsilon$ and an integer $r$ st for sufficiently large $l, k$. if
$$F: \begin{bmatrix} k \\ l \end{bmatrix}_2 \to \mathbb{F}_2^l \quad \text{satisfies}$$
$$\Pr\left[\, Test^F_{Gr,lin} \text{ acc} \right] \geq \delta$$
then $\exists$ $A \subseteq B$ st $\dim(A) + \text{co-dim}(B) \leq r$
& a global linear fn $f$
$$\Pr_{A \subseteq L \subseteq B}\left[\, F[L] = f|_L \right] \geq \varepsilon$$

Compose this Gr-test w/ outer object to obtain gap-2-to-2-games instance.

$\boxtimes$